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NUCLEAR PHYSICS

MONTHLY JOURNAL DEVOTED TO THE EXPERIMENTAL AND THEORETICAL STUDY OF ATOMIC NUCLEI, NUCLEAR FIELDS AND THE FUNDAMENTAL ASPECTS OF COSMIC RADIATION

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E. KONDAIAH, C. BADRINATHAN and K. V. K. IYENGAR, Interaction of fast neutrons with fluorine 561

J. F. VERVIER, Section efficace de capture radiative pour des neutrons d'une source Sb—Be 569

TH. A. J. MARIS, Quasi-free nucleon-nucleon scattering 577

ABDUS SALAM and J. TIOMNO, On the masses of elementary particles 585

A. UHLMANN, Schema einer Quantenmechanik mit indefiniter Metrik 588

B. F. BAYMAN and A. BOHR, On the connection between the cluster model and the SU_3 coupling scheme for particles in a harmonic oscillator potential . . 596

MARGARET E. LAW, G. W. HUTCHINSON and D. H. WHITE, Total cross sections for 910 MeV protons 600

T. H. R. SKYRME, The effective nuclear potential 615

T. H. R. SKYRME, The spin-orbit interaction in nuclei 635

T. H. R. SKYRME, Some distortion effects in the nuclear p-shell 641

GENTARO ARAKI, Quantal derivation of mass defect 650

CHOU KUANG-CHAO, A note on the decay of the Σ-hyperon and its antiparticle 652

V. G. SOLOVIEV, On the superfluid state of the atomic nucleus 655

B. S. DŽELEPOW, YU. V. KHOLNOV and V. P. PRIKHODTSEVA, A $0^+ \rightarrow 0^-$ transition in Ce^{140} 665

(continued on p. 3 of cover)

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INTERACTION OF FAST NEUTRONS WITH FLUORINE

E. KONDAIAH, C. BADRINATHAN and K. V. K. IYENGAR †

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Received 31 October 1958

Abstract: Fluorine targets were irradiated with fast neutrons arising from D(d, n), Be(d, n) and T(d, n) reactions using the deuterons accelerated in a one million volt Cockroft-Walton accelerator. Resulting charged particle spectra and low energy gamma ray spectra were studied. Using 15 MeV neutrons, the charged particle spectrum showed a small hump at (10.6±0.8) MeV which is consistent with a 10.5 MeV alpha group leading to the ground state of N^{16} in $F^{19}(n, \alpha)N^{16}$ reaction. The gamma spectra showed mainly 110±8 keV and 195±10 keV gamma rays. Relative intensities of these gamma rays at each of the neutron energies are given. Gamma rays of 300 keV and 390 keV energy that might arise in $F^{19}(n, \alpha)N^{16}$ reaction were looked for but not observed. Upper limits on their intensities with respect to 110 keV gamma rays are given. No real coincidence effect was found between 110 keV and 195 keV gamma rays up to 10 μ sec delay.

1. Introduction

Interaction of fast neutrons with fluorine can give rise to the following reactions [1]):

$$F^{19}+n \rightarrow F^{19*}+n'$$
$$N^{16}+He^4 - 1.49 \text{ MeV}$$
$$O^{19}+H^1 - 3.999 \text{ MeV}$$
$$O^{18}+H^2 - 5.8 \text{ MeV}$$
$$F^{18}+n+n-10.408 \text{ MeV}.$$

Day [2]), and Van Loef and Lind [3]) studied the gamma rays arising from $F^{19}(n, n')F^{19*}$ reaction using mono-energetic neutrons up to 2.56 MeV energy. The excitation function for $F^{19}(n, \alpha)$ reaction was studied by Marion and Brugger [4]) and was found to be zero up to 3.1 MeV neutron energy even though the threshold based on mass values is 1.49 MeV. Therefore, gamma rays from N^{16*} levels are expected to occur only for neutron energies above 3.1 MeV. James *et al.*[5]) found that $F^{19}(n, \alpha)N^{16}$ reaction leads to the 0.3 MeV state of N^{16} to a large extent and the alpha group leading to the ground state of N^{16}, if present, is only 4 % of the alpha group leading to the 300 keV excited state in N^{16}; they employed neutrons of 4.87 MeV energy. Marion and Brugger [4]) also found that the excitation function for $F^{19}(n, p)O^{19}$ reaction is zero up to 4.6 MeV neutron energy even though the threshold

† At present on study leave at Brown University, Providence, Rhode Island, U.S.A.

based on mass values is for this reaction 3.999 MeV. This difference between thresholds for (n, α) and (n, p) reactions based on mass values and the experimental threshold values is due to the finite barrier presented by nuclei for the emission of charged particles.

In the studies reported here, we used a thin (0.8 mm) CaF_2 crystal as both target and detector and analysed the pulse height spectrum of the resulting charged particles with a single channel analyser. For the gamma ray studies, CaF_2 crystal was replaced by a teflon cylinder (2″ diameter 3″ long) and the resulting gamma ray spectra were studied up to a gamma ray energy of 400 keV using a NaI(Tl) detector.

2. Experimental Arrangement

Fig. 1 gives the experimental arrangement used in the study of charged particle spectrum. A thin crystal of CaF_2 was used to cut down the electron pulses due to background gamma rays.

While studying gamma rays with NaI(Tl) detector, a more effective shield has become necessary to reduce the build up of iodine activity in the detector. Consequently, the tank of boric acid solution shown in fig. 1 was replaced by a thicker (24″ diameter, 18″ long) block of paraffin into which boric acid (45 lbs) was uniformly mixed. The general arrangement shown in fig. 2 was similar to Van Loef's [3]) arrangement.

2.1. NEUTRON MONITOR

Neutrons were continuously monitored with a monitor insensitive to gamma rays. The monitor scintillator was made † of a mixture of glyceryl borate plastic and zinc sulphide. This scintillator was placed at the centre of a paraffin block of dimensions 12″ by 12″ by 17″ which was kept at a height of 35″ from the ground level (fig. 2). The monitor made an angle of 45° with the deuteron beam of the accelerator.

Gamma insensitiveness of the monitor was tested in two ways. A 10 millicurie Co^{60} source placed directly on this scintillator did not give any count at the discriminator level used. Secondly, the monitor count remained constant to within 10 % when it was surrounded by no, one, two and three layers of lead of thickness $1\frac{1}{4}$″ each and the D(d, n), Be(d, n), or T(d, n) neutron source was put on.

There was nearly 22″ of paraffin interposed between neutron source and monitor scintillator. This thickness of paraffin is sufficient to reduce the energy of source neutrons (MeV) to slow (≈ 1 keV) neutrons, in spite of large energy differences in the primary neutron sources employed in these studies.

† We are grateful to Dr P. Radhakrishna for preparing this.

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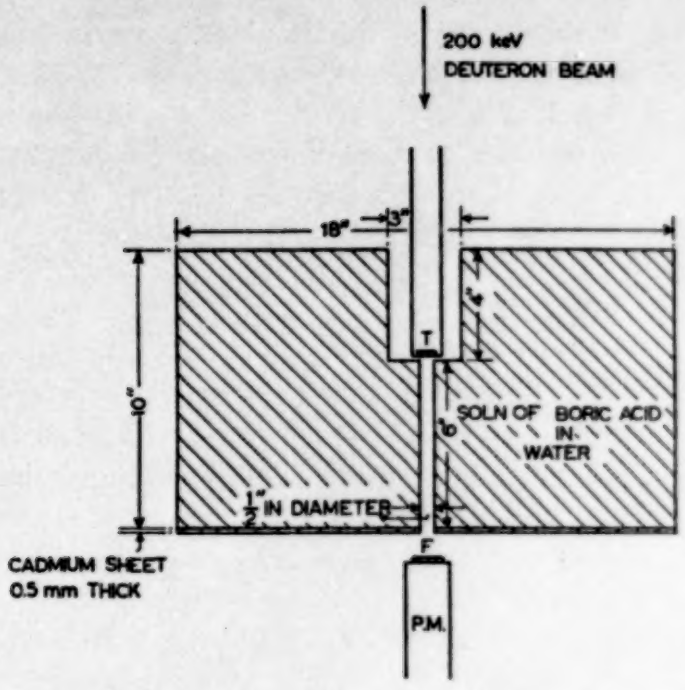

Fig. 1. Experimental arrangement for the study of charged particles. T. Titanium tritium target. F. CaF_2 crystal 0.8 mm thick and 2.5 cm diameter. P. M. RCA 5819 photomultiplier.

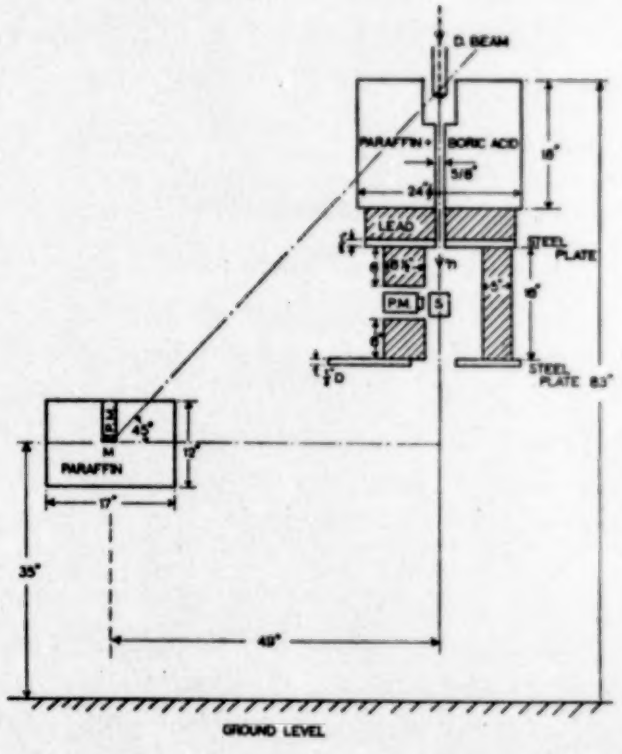

Fig. 2. Experimental arrangement for the study of gamma rays from (F^{19}+n) reactions. M. Neutron monitor. S. Teflon target (2" diameter, 3" long). N. Collimated neutron beam.

2.2. NEUTRON SOURCES

Deuterium and tritium absorbed in titanium, were used as targets † for D(d, n) and T(d, n) sources; a thick Be block was used as target for Be(d, n) neutron source. Deuterium targets were irradiated with 725 keV deuterons; tritium targets were worked with 200 keV deuterons; Be was bombarded with 800 keV deuterons. The neutrons incident on the fluorine scatterer, denoted by S in fig. 2, have the following energies [6]): D(d, n) source 3.8 MeV; T(d, n) source 15.1 MeV; Be(d, n) neutrons are polyergic as they lead to several excited states in B^{10}.

As the neutron collimator was loaded with boric acid and was lined with Cd, it is assumed that any neutrons from the source that enter the collimator body would not get mixed up with 0° fast neutrons which had a direct passage from the source through collimator hole (fig. 2) to the fluorine scatterer.

2.3. GAMMA DETECTOR:

A NaI(Tl) crystal of $1\frac{1}{2}''$ diameter and 1″ height was used for recording most of the gamma ray spectra in order to obtain high efficiency of detection. A few spectra were taken with a NaI(Tl) crystal of 1.75″ diameter and $\frac{1}{4}''$ height which gave much cleaner spectra. For the purpose of intensity estimates, the thick crystal data were used. Each experiment on the gamma ray spectrum consisted of two runs, one with teflon cylinder (2″ diameter and 3″ height) in position as scatterer and another replacing the teflon cylinder by a graphite cylinder containing the same number of carbon atoms as the teflon cylinder. The difference between these two spectra was taken to be due to fluorine.

3. Charged Particle Spectrum

The charged particle spectrum taken with 0.8 mm CaF_2 crystal using T(d, n) neutron source, and the polonium alpha spectrum taken with the same crystal are shown in fig. 3. The charged particle spectrum consists of protons, deuterons and alpha particles arising from fast neutron interactions with fluorine. The photomultiplier voltage and amplification of the amplifier were so chosen that electron pulses due to gamma rays were pushed out of the scale. The window width of the single channel analyser was kept at 1 volt. All the observations in the charged particle spectrum were measured to a statistical accuracy better than 3.5 %. The spectrum shows a small hump at (29 ± 1) volts channel threshold. The polonium alpha peak is seen to occur at 14.5 volts on the same scale. Assuming linearity of scale in this region, the 29 volt hump corresponds to (10.6 ± 0.8) MeV alpha energy which is consistent with the ground state alphas of 10.5 MeV to be expected

† These targets were supplied by A.E.R.E., Harwell, England.

from $F^{19}(n, \alpha)N^{16}$ reaction at this neutron energy. Apart from this small hump in the charged particle spectrum, we did not observe any other peak of comparable intensity. As the resolution of CaF_2 crystal for polonium alpha particles is found to be $\lesssim 40\,\%$, we do not expect to resolve close

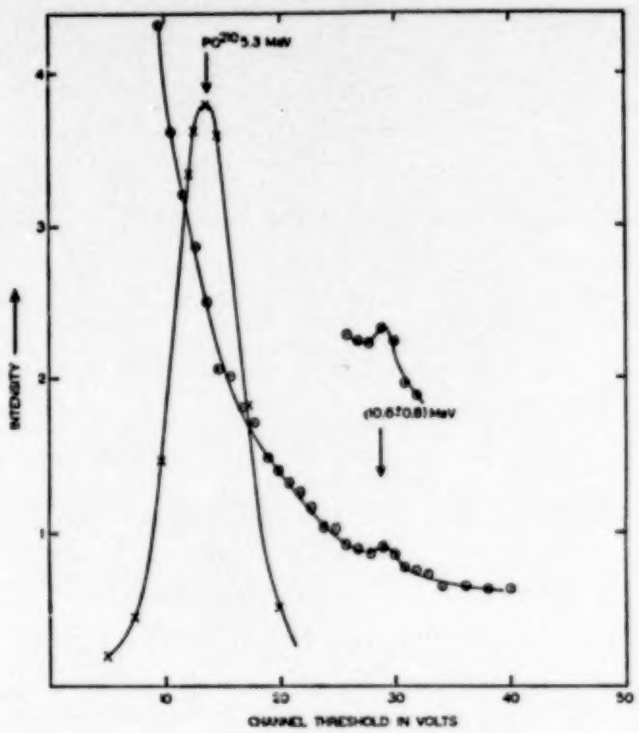

Fig. 3. Spectrum of charged particles.

x——x Po^{210}.

⊙——⊙ Spectrum of charged particles due of to $(F^{19}+n)$ reactions.

levels of the resulting nuclei in the charged particle spectrum. However, alpha groups leading to any excited states well above the ground state of N^{16} would have been detected if their intensity is comparable to the ground state alpha group.

4. Gamma Spectrum

The gamma spectrum obtained with a $\frac{1}{4}''$ thick NaI detector using T(d, n) neutron source is given in fig. 4. In spite of using a graphite cylinder containing the same number of carbon atoms as the teflon cylinder for background runs, some of the difference spectra showed the presence of 31 and 62 keV gamma rays due to iodine. The difference spectrum shown in fig. 4 gives clearly the two gamma rays at 110 ± 8 keV and 195 ± 10 keV energy. These gamma rays are due to $F^{19}(n, n')F^{19}$ reaction as they occur at all the neutron energies. No gamma rays of either 300 keV or 390 keV energy were observed. The gamma spectra obtained with D(d, n) and Be(d, n) neutrons were

similar to the one presented in fig. 4, showing only the 110 keV and 195 keV gamma rays. From the positive counting rates in the difference spectra at the positions where 300 and 390 keV if present should occur, one can give an

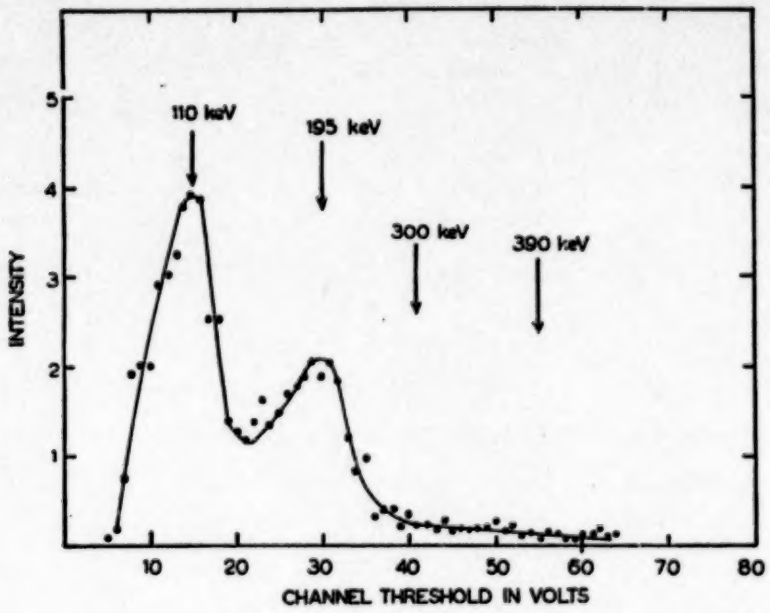

Fig. 4. γ-spectrum.

upper limit to the intensity of these gamma rays with respect to the 110 keV gamma rays.

Table 1 gives the relative intensities of 195 keV, 300 keV and 390 keV gamma rays with respect to 110 keV gamma ray at different neutron energies. From the last column of this table, one can get an idea of the higher limit on the intensities of 300 and 390 keV gamma rays with respect to

TABLE 1

Relative intensities of gamma rays

Neutron source	Gamma energy (keV)	Ratio of photopeak areas A_γ/A_{110}	Ratio of efficiencies E_{110}/E_γ	Ratio of intensities I_γ/I_{110}
D(d, n)	195	0.8 $\pm$0.08	1.2$\pm$0.15	0.76
	(300)	$<$ 0.05$\pm$0.02	1.6$\pm$0.3	$<$ 0.06
	(390)	—	1.9$\pm$0.5	—
Be(d, n)	195	0.78$\pm$0.24	1.2$\pm$0.15	0.74
	(300)	$<$ 0.1 $\pm$0.03	1.6$\pm$0.3	$<$ 0.11
	(390)	—	1.9$\pm$0.5	—
T(d, n)	195	1.0 $\pm$0.25	1.2$\pm$0.15	0.95
	(300)	$<$ 0.05$\pm$0.05	1.6$\pm$0.3	$<$ 0.06
	(390)	$<$ 0.02$\pm$0.02	1.9$\pm$0.5	$<$ 0.02

110 keV gamma ray at each of the neutron energies used. The errors given in column 3 of this table were estimated on the basis of consistency between different gamma spectra taken with the same neutron source. Errors given in column 4 refer to possible edge effects in NaI for the different gamma rays. Last column of the table gives the relative intensities after correcting for absorption of the gamma rays inside teflon source.

5. Coincidence Studies

The gamma spectra mentioned above did not show evidence for any direct transition gamma rays of 300 or 390 keV, from excited states of N^{16}, that might possibly have been formed in the $F^{19}(n, \alpha)N^{16}$ reaction [5]). In order to test whether these states of N^{16} decay by cascade emission through the first excited level in N^{16} at 118 keV, coincidence studies were undertaken. It is known [7]) that the 118 keV level of N^{16} has a half-life of 6.7 μsec. So, coincidences between 110 keV and 195 keV gamma rays were studied using a variable delay in the range 0 to 10 μsec. The variable delay has a large attenuation and consequently it became necessary to introduce univibrators to give a constant and sufficiently large pulse to trigger the coincidence circuit. The overall resolution time of the coincidence circuit used was (2.4 ± 0.1) μsec and was constant over the whole range of delays used. This was tested by using two independent sources and measuring the chance coincidences as well as singles in each channel for different delays introduced. From these chance coincidences and singles, the resolution time τ of the coincidence circuit was found.

Two NaI(Tl) crystals of $1\frac{1}{2}''$ diameter and $1''$ height were used facing each other as gamma detectors in the two channels. The window widths of the single channel analysers were adjusted to 25 keV so that large counting rates were obtained and any small drifts (< 10 keV) in the photo-peaks of 110 and 195 keV gamma rays might not affect the coincidence rate. The energy calibration of the single channel analysers was periodically tested and found to be consistent within 10 keV in the region 100 to 200 keV. The coincidence efficiency of the circuit was adjusted to 100 %.

Coincidences as well as singles were recorded simultaneously. Coincidence studies were made with teflon as well as graphite cylinder mentioned above for all the delay positions. Coincidence rates obtained in the case of teflon as well as graphite agreed with chance coincidence rates within 10 % for all positions of delay. The differences between teflon and graphite cases were not significantly different from zero. The statistical accuracy of these observations was better than 3 %. Coincidence experiments were done with both Be(d, n) and T(d, n) neutron sources. In the first case, it has also been found that 195 keV gammas were not in prompt coincidence with any gamma rays below 400 keV energy.

6. Results

From the studies reported here the following results were obtained:

(1) The charged particle spectrum showed a small hump at 10.6 ± 0.8 MeV alpha energy which agrees with the energy of the ground state alpha group in $F^{19}(n, \alpha)N^{16}$ reaction.

(2) The ratio of intensities of 195 keV to 110 keV gammas is about 1 for all the neutron energies used.

(3) The intensity of 300 keV gamma ray, if present, is $< 10\,\%$ of the 110 keV gammas for the neutron energies used.

(4) 110 keV and 195 keV gammas are not coincident within 10 μsec.

7. Discussion

The absence of other charged particle groups of comparable intensity to the 10.6 MeV alpha group shows that no other levels of N^{16} than the ground level and or the low lying levels close by to the ground level are involved in a prominant way in the $F^{19}(n, \alpha)N^{16}$ reaction at 15 MeV neutron energy.

Day [2]) found that the ratio of intensities of 195 keV to 110 keV gammas is 2.8 with a maximum error of 0.8 at a neutron energy 2.56 MeV. This ratio is nearly unity in our case presumably owing to a contribution from 118 keV gammas from de-excitation of the first level in N^{16} that might have been formed in the $F^{19}(n, \alpha)N^{16}$ reaction; this reaction has resonances at neutron energies 3.6 to 3.8 MeV [4]), which are covered by our D(d, n) neutron energy.

The gamma spectra as well as the coincidence studies show that the 300 and 390 keV levels of N^{16} are not excited in the $F^{19}(n, \alpha)N^{16}$ reaction at the neutron energies used. It appears that this reaction goes mainly to the ground state and or to the first excited state in N^{16} at the neutron energies used in the present work.

Messrs. M. D. Deshpande, U. T. Raheja and C. S. Somanathan have given technical assistance during the course of this work.

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SECTION EFFICACE DE CAPTURE RADIATIVE POUR DES NEUTRONS D'UNE SOURCE Sb-Be

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Centre de Physique Nucléaire de l'Université de Louvain

Reçu le 31 Octobre 1958

Abstract: Radiative capture cross sections of 7 isotopes for 25-keV Sb-Be photoneutrons have been measured by comparison with the corresponding thermal neutron cross sections; the agreement with previous measurements is satisfactory.

1. Introduction

La détermination des sections efficaces de capture radiative pour neutrons intermédiaires et rapides présente un intérêt considérable pour les théories cosmologiques [1]) et nucléaires, et sont d'une grande importance pratique pour le calcul des réacteurs nucléaires intermédiaires et rapides. Un certain nombre de résultats ont déjà été obtenus dans ce domaine depuis quelques années [2]).

En particulier, la section efficace de capture radiative d'un grand nombre d'isotopes pour des neutrons de 25 keV, obtenus avec une source de photoneutrons Sb—Be, a été mesurée à trois reprises par des méthodes différentes. Hummel *et al.*[3]) ont déterminé le rapport σ_{xp}/σ_{xt} entre les sections efficaces de capture radiative de l'isotope x pour des photoneutrons de la source Sb—Be (σ_{xp}) et pour des neutrons thermiques (σ_{xt}); le flux de photoneutrons était mesuré par comparaison avec une source Ra—Be standard. Macklin *et al.*[4]) ont utilisé une technique de comptage absolu des rayons γ. Booth *et al.*[5]) ont mesuré le rapport σ_{xp}/σ_{xt} avec I^{127} comme isotope de référence. Les résultats de Macklin et de Booth concordent généralement entre eux (sauf pour Mo^{98} et Sb^{123}) mais sont le plus souvent en désaccord avec ceux de Hummel.

Nous avons mesuré la section efficace de capture radiative de 7 isotopes par la même méthode que Booth, mais en utilisant, comme isotopes de référence, In^{115} et Au^{197}, pour lesquels σ_{xt} est connu avec plus de précision que pour I^{127}. Nous avons choisi, pour σ_{xp} (x = In^{115} et Au^{197}), la moyenne pondérée des résultats de Macklin et de Booth qui concordent entre eux; cette moyenne se situe d'ailleurs dans le prolongement des résultats obtenus pour des énergies plus élevées de neutrons [2]), mieux que les valeurs de

† Boursier du Syndicat d'Etude de l'Energie Nucléaire de Belgique (Belgonucléaire).

Macklin et de Booth, considérées séparément. Nous obtenons de la sorte deux valeurs du rapport ϕ_t/ϕ_p entre les flux de neutrons thermiques (ϕ_t) et de photoneutrons (ϕ_p), l'une avec In^{115}, l'autre avec Au^{197}; nous choisissons alors, comme rapport ϕ_t/ϕ_p, la moyenne pondérée de ces deux résultats.

2. Description de l'Appareillage

2.1. SOURCE DE PHOTONEUTRONS Sb—Be

La source de photoneutrons Sb—Be est représentée à la figure 1. L'antimoine a été irradié pendant 2 mois au réacteur BRI de Mol (Belgique) et

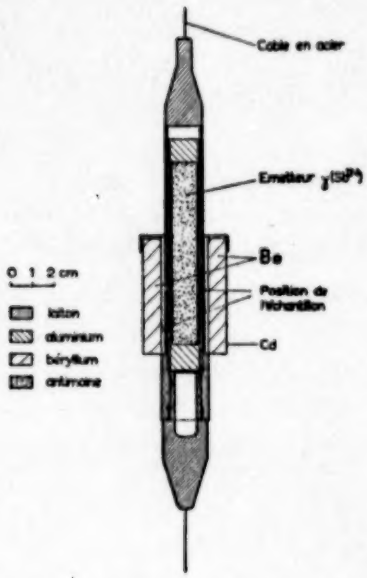

Fig. 1. Source de photoneutrons Sb—Be.

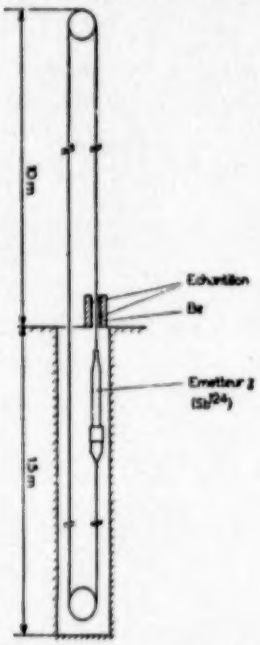

Fig. 2. Géométrie des irradiations avec les photoneutrons de la source Sb—Be. Entre les irradiations, l'émetteur γ se trouve à 1.50 m sous terre. Pour effectuer une irradiation, on fait monter au moyen d'un moteur qui actionne la poulie inférieure, l'émetteur γ qui prend au passage le cylindre en béryllium et l'échantillon et s'élève à 10 m au dessus du sol où a lieu l'irradiation.

l'activité du Sb^{124} à la fin de l'irradiation était d'environ 10 curies. Les irradiations avec la source Sb—Be se font à l'extérieur, à 10 m du sol (fig. 2), pour éviter au maximum les neutrons thermiques et de résonances produits par le ralentissement des photoneutrons dans le sol et sur les murs du laboratoire; l'ensemble de la source SB—Be est d'ailleurs entouré d'une protection supplémentaire en cadmium (fig. 1). Les échantillons métalliques, en forme de tubes cylindriques de 5 cm de long et 2 cm de diamètre extérieur, sont insérés entre l'émetteur γ et le cylindre en béryllium (fig. 1); cette géométrie permet d'utiliser au maximum le flux disponible de photoneutrons.

2.2. SOURCE DE NEUTRONS THERMIQUES

La source de neutrons thermiques est constituée par 2.4 curies de Ra—Be entourés de paraffine; la géométrie des irradiations est représentée à la

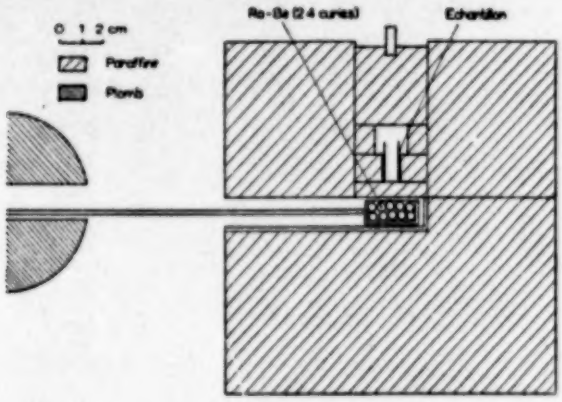

Fig. 3. Géométrie des irradiations avec des neutrons thermiques.

fig. 3. Deux irradiations sont effectuées pour chaque échantillon: pour l'une, l'échantillon est découvert, comme représenté à la fig. 3, pour l'autre, il est contenu dans un cylindre en cadmium dont les parois ont 0.5 mm d'épaisseur; la différence entre les deux activités représente la contribution des neutrons thermiques.

2.3. SYSTÈME DE COMPTAGE

L'activité des échantillons irradiés est mesurée avec un compteur GM (Type: 20th Century, B6. Temps mort, déterminé par la méthode des deux sources: 112 ± 4 μs) à parois minces (30 mg/cm^2); l'installation de comptage que nous avons utilisée est représentée à la fig. 4.

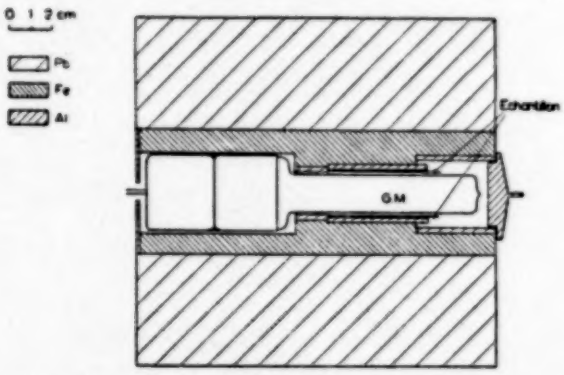

Fig. 4. Géométrie du système de comptage.

3. Méthode Expérimentale

3.1. MESURES DES RAPPORTS D'ACTIVITÉS

On mesure les rapports R_{xp}/R_{xt} entre les activités obtenues après irradiation de l'isotope x aux photoneutrons (R_{xp}) et aux neutrons thermiques

(R_{xt}), celle-ci étant corrigée de l'activité due aux neutrons de résonances obtenue après l'irradiation effectuée sous cadmium. Ces trois irradiations sont réalisées avec le même échantillon et dans les mêmes conditions (durée de l'irradiation, intervalle de temps entre la fin de l'irradiation et le début du comptage, durée du comptage, géométrie du comptage, etc.), ce qui élimine les corrections de géométrie, d'auto-absorption des β dans l'échantillon, de rétrodiffusion, de temps de vie, etc., et par conséquent les erreurs inhérentes à ces corrections.

3.2. CORRECTION D'AUTO-ABSORPTION DES NEUTRONS THERMIQUES

Lorsqu'on mesure les rapports d'activité R_{xp}/R_{xt}, on doit apporter aux résultats une correction due à l'auto-absorption des neutrons thermiques et de résonances dans l'échantillon; celle-ci produit une dépression du flux au voisinage de la surface intérieure de l'échantillon, dont l'activité β est comptée. Pour déterminer expérimentalement cette correction, nous avons mesuré le rapport R_{xp}/R_{xt} en fonction de l'épaisseur d de l'échantillon pour des éléments dont la section efficace pour neutrons thermiques est appréciable (Ag, In et Au); les résultats sont repris aux figs. 5, 6 et 7: les courbes de R_{xp}/R_{xt} en fonction de d sont des droites:

$$[R_{xp}/R_{xt}](d) = [R_{xp}/R_{xt}](0) \times (1+C_x d), \qquad (1)$$

où $C_{Ag} = 1.7\ \text{mm}^{-1}$, $C_{In} = 4.8\ \text{mm}^{-1}$, $C_{Au} = 2.26\ \text{mm}^{-1}$.

La valeur $[R_{xp}/R_{xt}](0)$ de R_{xp}/R_{xt} compte tenu de la correction d'auto-absorption se calcule à partir de l'équation (1) avec les coefficients C_x que nous avons obtenus; cette correction est négligeable pour les isotopes que nous avons étudiés autres que Ag^{109}, In^{115} et Au^{197}. La fig. 8 représente, en fonction de d, les différentes activités obtenues en irradiant de l'argent aux photoneutrons (courbe p), aux neutrons thermiques sans cadmium (tr), avec cadmium (r) et la différence (t) entre les courbes tr et r. La dépression du flux due à l'auto-absorption des neutrons dans l'échantillon explique la diminution de l'activité due aux neutrons thermiques et de résonances pour $d > 0.2$ mm; d'autre part, pour $d < 0.2$ mm, l'épaisseur d de l'échantillon est de l'ordre de grandeur de la demi-épaisseur $d_{\frac{1}{2}}$ pour l'absorption des β dans l'argent et les activités obtenues augmentent avec d. Ces deux facteurs jouent en sens inverses et permettent d'expliquer qualitativement les courbes de la fig. 8.

3.3. MESURE DU RAPPORT DES FLUX DE NEUTRONS

On détermine le rapport ϕ_t/ϕ_p entre les flux de neutrons thermiques et de photoneutrons en mesurant les rapports R_{Int}/R_{Inp} et R_{Aut}/R_{Aup} entre les activités de In^{116} (54 min) et Au^{198} (2.72 d); on a alors

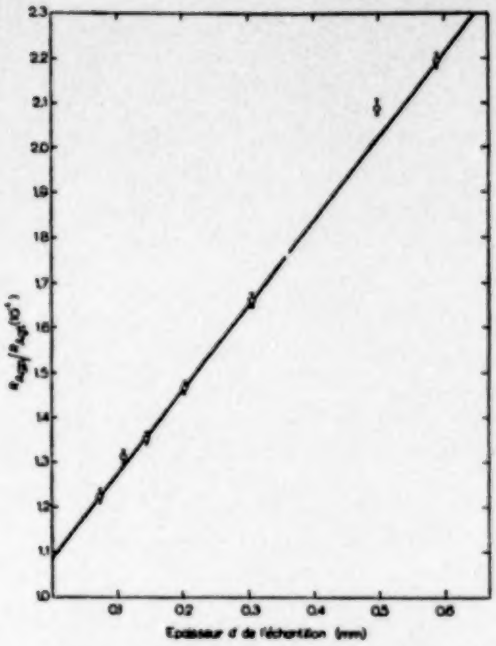

Fig. 5. Rapport des activités R_{xp}/R_{xt} en fonction de l'épaisseur d de l'échantillon pour Ag.

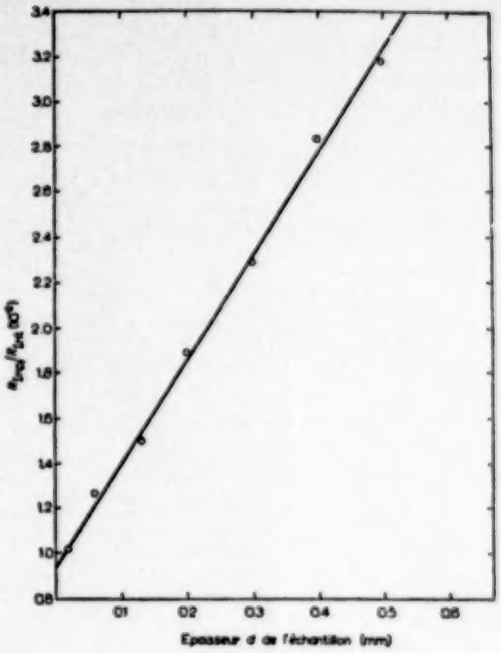

Fig. 6. Rapport des activités R_{xp}/R_{xt} en fonction de d pour In.

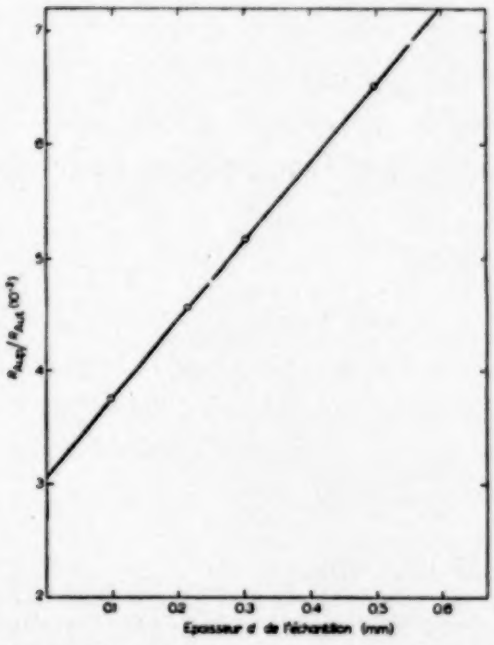

Fig. 7. Rapport des activités R_{xp}/R_{xt} en fonction de d pour Au.

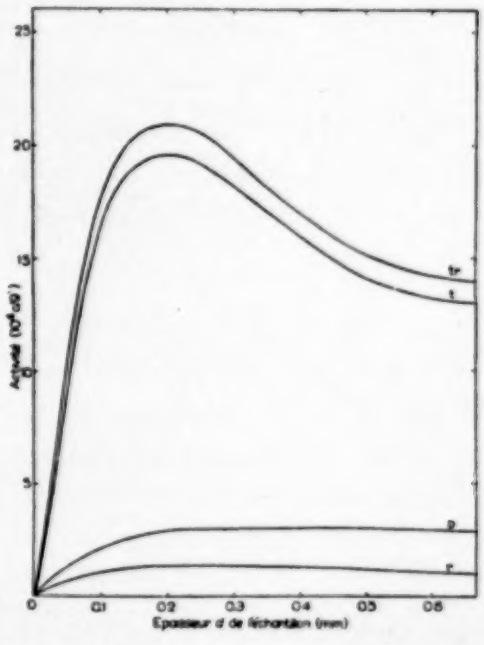

Fig. 8. Activités, en fonction de l'épaisseur d de l'échantillon, obtenues en irradiant de l'argent aux photoneutrons (courbe p), aux neutrons thermiques sans cadmium (tr), avec cadmium (r), et la différence (t) entre les courbes tr et r.

$$\frac{\phi_t}{\phi_p} = \frac{R_{Int}}{R_{Inp}} \times \frac{\sigma_{Inp}}{\sigma_{Int}} = \frac{R_{Aut}}{R_{Aup}} \times \frac{\sigma_{Aup}}{\sigma_{Aut}}. \qquad (2)$$

Les valeurs de σ_{Inp}, σ_{Int}, σ_{Aup} et σ_{Aut} que nous avons choisies sont données à la table 1. Nous choisissons alors, pour ϕ_t/ϕ_p, la moyenne pondérée des 2 résultats obtenus avec In et Au, qui concordent d'ailleurs entre eux aux erreurs statistiques près; les valeurs ainsi obtenues sont données à la table 1.

TABLE 1

Calcul du rapport ϕ_t/ϕ_p entre les flux de neutrons thermiques et de photoneutrons à partir de l'équation (2). ϕ_p est normalisé au même jour, compte tenu de la décroissance de l'antimoine Sb^{124}

Isotope de référence	In^{115}(54 min)	Au^{197}(2.72 d)
σ_{xt} (b) [2])	155 ± 10	98.8 ± 0.3
σ_{xp} (mb) Macklin *et al.* [4])	805 ± 80	1120 ±110
Booth *et al.* [5])	980 ±220	890 ±190
moyenne pondérée	825 ± 75	1065 ± 95
$\sigma_{xp}/\sigma_{xt}(10^{-2})$	0.532± 11 %	1.08 ± 9 %
R_{xt}/R_{xp} normalisé	55.1 ± 1 %	24.2 ± 1 %
ϕ_t/ϕ_p normalisé	0.293± 0.032	0.262± 0.024
moyenne pondérée	0.272±0.020(7.35 %)	

3.4. MESURE DU RAPPORT DES SECTIONS EFFICACES

On détermine le rapport σ_{xp}/σ_{xt} entre les sections efficaces de capture de l'isotope x pour des photoneutrons et des neutrons thermiques en mesurant le rapport des activités R_{xp}/R_{xt}; on a alors

$$\frac{\sigma_{xp}}{\sigma_{xt}} = \frac{R_{xp}}{R_{xt}} \times \frac{\phi_t}{\phi_p}. \qquad (3)$$

La valeur de ϕ_t/ϕ_p est obtenue par les mesures précédentes, compte tenu de la diminution de ϕ_p due à la décroissance de l'activité du Sb^{124} avec un temps de vie de 60.9±0.2 jours [6]). Les valeurs des sections efficaces thermiques σ_{xt} sont celles de la compilation de Hughes *et al.* [2]).

4. Résultats expérimentaux

Nos résultats expérimentaux sont donnés à la table 2 en même temps que ceux obtenus précédemment par d'autres expérimentateurs [3,4,5]); notons que les valeurs de Hummel *et al.*[3]), données par rapport aux sections efficaces „naturelles" pour neutrons thermiques de Seren *et al.*[7]), ont été transformées en sections efficaces isotopiques par rapport aux valeurs plus récentes des sections efficaces thermiques [2]).

TABLE 2

Sections efficaces de capture radiative pour des neutrons d'une source Sb–Be.

Isotope	$\frac{R_{xp}}{R_{xt}}$ (10^{-2}) ± erreur (%)	$\frac{\phi_t}{\phi_p}$ ±7.35 %	σ_{xt}(b) ± $\Delta\sigma_{xt}$(%)	$\sigma_{xp} \pm \Delta\sigma_{xp}$(mb)	Autres résultats	Référence
Al^{27}	2 4	0.562	0.23 2	2.6 ± 0.22	1.75 ± 0.44 1.4 ± 0.2	3) 4)
Ti^{50}	3.87 9	0.417	0.14 22	2.3 ± 0.6	5 ± 3	4)
Cu^{63}	4.27 1	0.570	4.3 5	105 ± 9.4	183 ± 47 116 ± 12 114 ± 24	3) 4) 5)
Cu^{65}	4.33 2	0.570	1.8 22	44 ± 10	64 ± 19 46 ± 5 48 ± 12	3) 4) 5)
Mo^{98}	3.35 4	0.275	0.45 22	415 ± 98	209 ± 21 390 ±120	4) 5)
Mo^{100}	2.33 2	0.275	0.20 25	148 ± 39	259 ±101 38 ± 8	3) 4)
Ag^{107}	10.95 1	0.355	45 9	1.788 ±208	2105 ±405 930 ±180	3) 4)

σ_{xp} est calculé à partir de l'équation (3) avec les valeurs de R_{xp}/R_{xt}, ϕ_t/ϕ_t et σ_{xt} indiquées dans la table. Les diverses valeurs de ϕ_t/ϕ_p tiennent compte de la décroissance de l'antimoine Sb^{124}; les valeurs de σ_{xt} sont celles de la compilation de Hughes *et al.* [2]). Sont également donnés les résultats d'autres expérimentateurs [3, 4, 5]).

Pour Ti^{50}, Cu^{63} et Cu^{65}, nos résultats concordent avec ceux des autres expérimentateurs. Notre valeur de σ_{Alp} est plus élevée que celles de Hummel et de Macklin *et al.* [4]), mais concorde mieux que celles-ci avec les mesures de Barschall *et al.*[8]) dans cette région d'énergie. Dans le cas de Mo^{98}, Mo^{100} et Ag^{107}, nos résultats sont en accord avec ceux de Hummel et de Booth *et al.*[5]) mais pas avec ceux de Macklin. On voit donc que nos mesures concordent généralement avec celles de Hummel et de Booth, qui ont utilisé une méthode fort semblable à la nôtre, mais que l'accord est moins bon avec celles de Macklin, dont la technique est assez différente; dans les cas où nos résultats ne concordent pas avec ceux de Macklin ceux-ci sont toujours plus petits que les nôtres: ces divergences pourraient donc s'expliquer par la différence entre les méthodes utilisées.

5. Conclusion

Nous avons mesuré la section efficace de capture radiative de 7 isotopes pour des neutrons de 25 keV obtenus avec une source Sb—Be; nos résultats concordent assez bien avec ceux d'autres expérimentateurs, les divergences pouvant s'expliquer par des différences de méthodes.

Nous remercions M. le professeur M. de Hemptinne et M. A. Martegani pour leurs conseils et suggestions, et le Syndicat d'Etude de l'Energie Nucléaire pour son aide financière au cours de ce travail; nous remercions également le Centre d'Etude de l'Energie Nucléaire, et spécialement M. M. Nève de Mévergnies, dont l'aide nous a été très précieuse pour la construction de la source de photoneutrons.

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QUASI-FREE NUCLEON-NUCLEON SCATTERING

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Abstract: A simple approximation is described for the calculation of quasi-free † proton-proton scattering using distorted incoming and outgoing waves. The method is applied to Li^7. The results show that in this light nucleus the refraction by the collective nuclear potential does not destroy the strong connection between the angular correlation of the emerging protons and the momentum distribution of the knocked-out proton in the nuclear shell concerned.

1. Introduction

In recent papers [1,2]) (quoted hereafter as I and II) it was theoretically and experimentally shown that quasi-free † nucleon-nucleon scattering can be used as a tool for the investigation of the structure of nuclei. For the planning and the interpretation of future experiments, it seems desirable to perform a more detailed calculation of the cross sections expected in a typical case. In particular, it has not yet become clear how far multiple scattering in the nucleus destroys the direct connection between the angular correlations of the emerging protons and the momentum distribution of the struck proton in the nuclear shell concerned.

The present note describes a simple approximation for a calculation with distorted waves. The method, which is a generalization of an approximation used by Squires [3]) for inelastic scattering, is applied to the case of Li^7.

In the following, the essential contents of I and II are assumed to be known, and the symbols not newly defined have the earlier meaning ††.

2. The Approximation Method

The cross section for quasi-free proton-proton scattering, averaged over the orientations of the bombarded nucleus and summed over the orientations of the residual one, using the impulse approximation with distorted incoming and outgoing waves, is given by [1])

† This is perhaps a clearer expression than the usual name "quasi-elastic" scattering, which recently has also been used for inelastic scattering with a small energy loss.

†† In the right hand side of eq. (1) of I, a factor 2 should be inserted.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}=\frac{2\pi}{\hbar}\cdot\frac{1}{v_0}\cdot\sum_{\mathrm{f}}\overline{\left|\sum_{m_s} g'_{\mathrm{f}lm_s}\cdot R_{12,03}\right|^2}\cdot\delta(E_1+E_2+E_n+E_\mathrm{b}-E_0) \quad (1)$$

For $g'_{\mathrm{f}lm}$ we take the relativistic generalization of its earlier definition:

$$g'_{\mathrm{f}lm}(\mathbf{k})=A^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\int\psi_\mathrm{f}^*(2,\ldots,A)\exp(-i\mathbf{k}\cdot\mathbf{r}_1)s_m^*(1)t_\mathrm{p}^*(1)\psi_\mathrm{i}(1,\ldots,A)$$
$$\cdot\exp\left[-i\sum_n\frac{E_n}{k_n\hbar^2c^2}\int V_n\mathrm{d}s_n\right]\mathrm{d}_1\ldots\mathrm{d}_A. \quad (2)$$

Here $h\mathbf{k}_n=\mathbf{p}_n$, iE_n/c^2 is the energy-momentum 4-vector of the proton with index n and the integrals in the exponent have to be taken along the classical path in the direction of $\mathbf{k}_n$ between infinity and the collision point. As in I (2) the horizontal bar in eq. (1) denotes the averaging over the orientations of the initial state. From these formulae the influence of the multiple scattering in the nucleus was in I roughly estimated by taking a constant average absorption and phase change for each shell involved, independent of the region in the nucleus where the collision takes place.

In the present note this approximation is improved by replacing the distorting exponential in eq. (2) by a non-constant analytic function of the coordinates of the collision point, which still allows a closed integration if harmonic oscillator wave functions are taken for the nuclear protons.

The distorting exponential is replaced by the function

$$D(x_1,x_2,x_3)=\exp[i(a+\sum_k b_kx_k+\sum_k\sum_l c_{kl}x_kx_l)][1-\exp(\alpha+\sum_k\beta_kx_k$$
$$+\sum_k\sum_l\gamma_{kl}x_kx_l)], \quad (3)$$

x_1, x_2 and x_3 representing the Cartesian x, y and z coordinates. The twenty real constants a, b_k, $c_{kl}(=c_{lk})$, d, β_k and $\gamma_{kl}(=\gamma_{lk})$ have to be suitably chosen and depend on the distorting potentials and on the momenta of the incoming and outgoing protons.

With this simple choice of a spin-independent distortion and harmonic oscillator wave functions, one may not hope to understand details of the experimental results, such as the deviation of the ratio of the $\mathrm{p}_{\frac{3}{2}}$- and $\mathrm{p}_{\frac{1}{2}}$-peaks in the oxygen spectrum [2]) from the value 2. But the distorting exponential is a smooth function and, apart from the spin dependence, one should be able to approximate it well inside of the nucleus by a suitably chosen $D(x_i)$. We therefore expect to find correctly the main modifications introduced by the use of distorted waves.

We first simplify and to a certain extent specialize eq. (1). In this expression the term $\sum_{m_s}g_{\mathrm{f}lm_s}R_{12,03}$ is equal to $(\sum_m|g'_{\mathrm{f}lm}|^2)^{\frac{1}{2}}$ times the matrix element for the cross section of the scattering process as considered hitherto, but in which the nuclear proton is replaced by a free proton with the spin wave function

$$\frac{\sum_m g'_{\mathrm{fl}m} s_m}{\sqrt{\sum_m |g'_{\mathrm{fl}m}|^2}}$$

and momentum $\hbar(\mathbf{k}_2+\mathbf{k}_1-\mathbf{k}_0)$. Quantizing the spin in the z-direction, the polarization vector of such a proton is given by

$$\mathbf{P}_{\mathrm{fl}} = \frac{\mathrm{Re}(g'^{*}_{\mathrm{fi}+} g'_{\mathrm{fl}-});\quad \mathrm{Im}(g'^{*}_{\mathrm{fi}+} g'_{\mathrm{fl}-});\quad |g'_{\mathrm{fl}+}|^2-|g'_{\mathrm{fl}-}|^2}{\sum_m |g'_{\mathrm{fl}m}|^2}. \tag{4}$$

Likewise,

$$\frac{\sum_{\mathrm{f}} \overline{|\sum_{m_3} g'_{\mathrm{fl}m_3} R_{12,03}|^2}}{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2}}$$

is reducible to the cross section for the scattering process in which the above-mentioned replacement is made for all orientations of the initial and final nuclear states. The "beam" replacing the nuclear proton is then defined by the polarization vector

$$\mathbf{P} = \frac{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2 \mathbf{P}_{\mathrm{fl}}}}{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2}}$$

$$= \frac{\mathrm{Re}\sum_{\mathrm{f}} \overline{(g'^{*}_{\mathrm{fi}+} g_{\mathrm{fl}-})};\quad \mathrm{Im}\sum_{\mathrm{f}} \overline{(g'^{*}_{\mathrm{fi}+} g'_{\mathrm{fl}-})};\quad \sum_{\mathrm{f}} \overline{|g'_{\mathrm{fl}+}|^2}-\sum_{\mathrm{f}} \overline{|g'_{\mathrm{fl}-}|^2}}{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2}} \tag{5}$$

and the momentum $\hbar(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0)$. In case there exists a plane of mirror symmetry, $\mathbf{P}$ is perpendicular to it.

From now on, the special case is considered in which the momenta of the emerging protons are chosen to be of equal magnitude and directed in one plane with and at equal angles to the momentum of the bombarding particle. This case is not only experimentally interesting, but also the calculations simplify considerably; and, independently of their details, some general conclusions can be drawn. The origin of the Cartesian coordinate system is taken in the centre of mass of the bombarded nucleus, the z-axis in the direction of the incoming protons and the x-axis in the scattering plane. Assuming unpolarized incoming protons and target nuclei and parity conservation, there are two planes of mirror symmetry, and consequently $\mathbf{P}$ must vanish.

In the appendix the following result of the elimination of $R_{12,03}$ from eq. (1) for this case is derived:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1 \mathrm{d}E_1 \mathrm{d}\Omega_2 \mathrm{d}E_2} = \frac{4k_1^2}{\hbar v_0} \frac{p_1^2 c^2 \sin^2\theta + m^2c^4}{2p_1^2 c^2 \sin^2\theta + m^2c^4} \frac{\mathrm{d}\sigma_{\mathrm{fr}}}{\mathrm{d}\Omega'} \sum_f \sum_m \overline{|g'_{flm}|^2} \tag{6}$$
$$\delta(E_1 + E_2 + E_n + E_b - E_0).$$

$\mathrm{d}\sigma_{\mathrm{fr}}/\mathrm{d}\Omega'$ is the differential cross section for free protons with energies equal to $\sqrt{[p_1^2 c^2 \sin^2\theta + m^2c^4]}$ and scattering angles of 90°; all these quantities are taken in the centre of mass system.

In eq. (6) the dependence of the angular correlations on the properties of the target nucleus is contained only in $\sum_l \sum_m \overline{|g'_{flm}|^2}$, which is calculable with the help of approximation (3). Because of the symmetries of the problem, only $a, b_3, c_{11}, c_{22}, c_{33}, \alpha, \beta_3, \gamma_{11}, \gamma_{22}$, and γ_{33} in $D(x_i)$ have non-vanishing values.

3. The Example of Li^7

The results of this symmetric case are now applied to Li^7 as the target nucleus. It is assumed that its p-proton is j-j-coupled, and that a clean knocking-out of this proton leads to one $j = 0$ state of He^6. In reality, these assumptions are probably far from fulfilled and He^6 will be left in various excited states. Experimentally, the contributions of these states, all caused by p-shell collisions, are difficult to resolve and we expect that the sum of their cross sections should be given quite well by the corresponding one of our pure j-j-coupling model.

The non-vanishing parameters in $D(x_i)$ have been determined by fitting $D(x_i)$ to the distorting exponential for square well potentials. The values are taken equal at the origin, at the intersections of the z- and x-axis with the surface of the potentials and in the points of the y-axis with y-coordinates equal to the root mean square of the y-coordinates of the nucleons in the nucleus.

Of course, this determination of the parameters in $D(x_i)$ is rather arbitrary. If, for example, D is adjusted to the distorting exponential caused by Gaussian potentials, $1-|D|$ becomes larger at the origin and its width smaller. For the moment we shall not investigate the sensitivity of the final result of the calculation to the choice of the adjustment procedure and which choice is the most reasonable. Therefore, only the general trends of the results obtained are expected to be reliable.

The value of the factor in front of the δ-function in eq. (6) is plotted in figures 1a and 1b for collisions in the s- and the p-shells in Li^7 with bombarding energies of 180 and 400 MeV. The mean free paths in nuclear matter have been chosen as 2, 4, and 4×10^{-13} cm, and the real parts of the distorting square well potentials as 20, 15 and 5 MeV at 100, 200 and 400 MeV respectively. The classical frequency of the harmonic oscillator potential generating the nuclear wave functions is taken to be

$$\hbar\omega = 21.3 \text{ MeV},$$

resulting in a root mean square nuclear radius of $1.3 \times 7^{\frac{1}{2}} \times 10^{-13}$ cm, which is equal to that of the homogeneous sphere chosen as the boundary of the square well potentials. No momentum dependence of the oscillator potential has been assumed in this example. For $d\sigma_{tr}/d\Omega'$ at 90°, we have taken 4mb · ster^{-1} at all energies.

The dashed curves in the figures have been calculated with the constant reduction factor of I. The qualitative features of these curves are easy to understand. At 45° scattering angles the knocked-out protons are nearly in

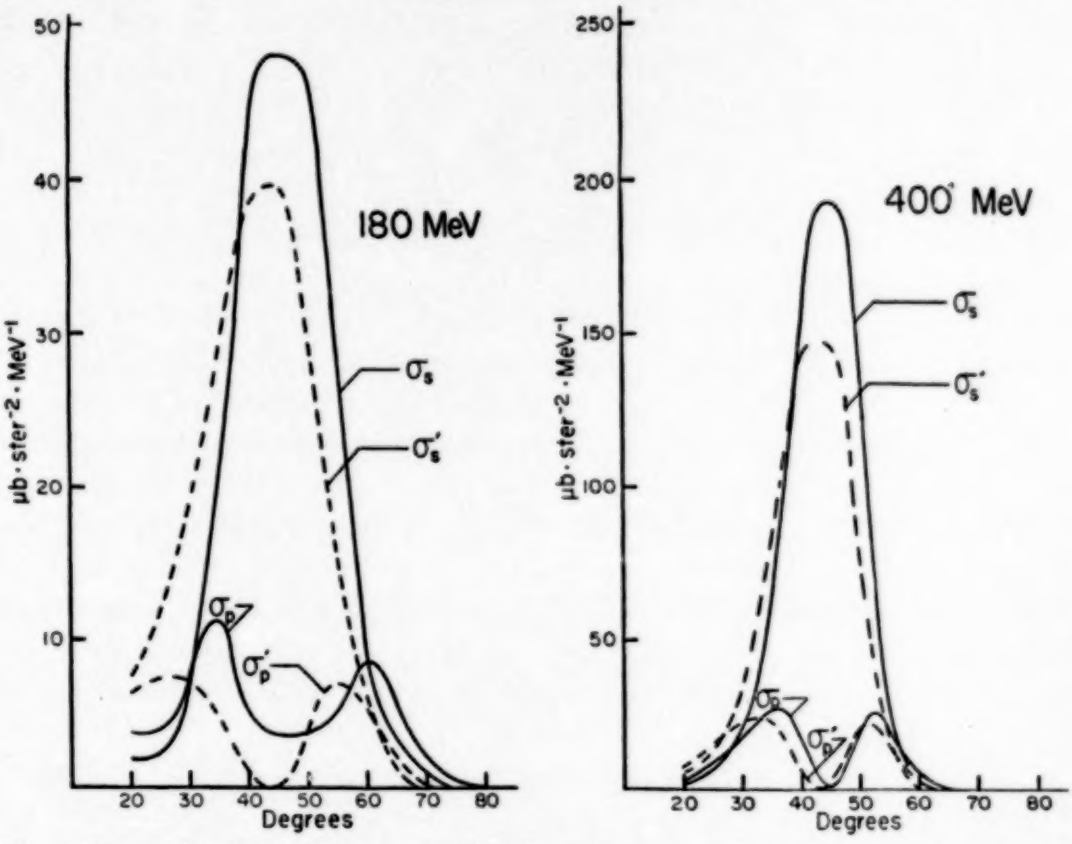

Fig. 1. The cross sections for symmetric coplanar quasi-free proton-proton scattering on Li^7 for 180 and 400 MeV bombarding energy, calculated with the present approximation method (full curves) and with a constant reduction factor (dashed curves).

rest in the nucleus. Here, the s-shell contribution has of course a maximum, and the one of the p-shell vanishes, its wave function in momentum space being antisymmetrical. The p-curve has maxima at angles roughly corresponding to the occurrence in the original nucleus of the most probable proton momenta opposite to, and in, the beam direction.

The full curves are based on the approximation described in the present paper. They show the same general behaviour as the dashed curves and the relation to the momentum distributions in the shells is still clear in a qualitative sense.

As remarked earlier the result of our example should not be taken too literally. It does seem likely that the approximations involved in the basic formulae (1) or (6), in the general form of (3), and even in the assumption of pure j-j coupling in Li^7 are sufficiently good for our present purpose of a rough quantitative description of the angular correlations. However, the determination of the parameters in $D(x_i)$ can certainly be improved considerably. Furthermore, the harmonic oscillator nuclear wave functions have too short tails at the nuclear surface, especially for the least bound particles. This will be of importance, because the strong nuclear absorption emphasizes the contributions of the collisions taking place near the nuclear surface. Also the amount of low momentum components must have been underestimated by taking harmonic oscillator wave functions. We hope to investigate these and related questions at a later time. It is to be expected that the p-shell results will be the most affected.

It is interesting to note that for the coplanar symmetric case only the p-shell particles contribute to $\sum_t \sum_m \overline{|g'_{tlm}|^2}$, whose wave functions contain the space wave function with orbital angular momentum component zero in the beam direction. The contributions of the other components vanish, because of the reflection symmetries of the situation. Consider now the absorptions along the classical paths of the bombarding and scattered protons for various collision points in the nucleus. Clearly, the collisions taking place on the largest distance from the z-axis, and especially those occurring farthest away from the scattering plane, result in the smallest total absorption. Because of the centrifugal force, the p-wave functions with angular momentum ± 1 along the beam direction have larger densities in these regions of the nucleus than the p-wave function with zero angular momentum z-component. This last function is in this respect quite equivalent to the s-wave function. Therefore, using oscillator functions, the intensity-increasing influence of the surface effect can only become strong for non-coplanar or asymmetric scattering, to which the other p-shell functions also contribute.

It is difficult to compare the experimental results of 180 MeV quasi-free proton-proton scattering on Li^7 as reported in II with the present calculations, because the experimental set-up allowed a large amount of non-coplanar correlation. The calculations indicate that measurements with improved angular resolution may be worthwhile, because the angular dependence of the correlation cross section seems to be markedly dependent on the momentum distributions in the shells concerned.

We thank Dr. Chr. Pfeufer for a most helpful discussion. The stimulating cooperation of Drs. Peter Hillman and H. Tyrén was essential for this note.

Appendix

The reduction of $|\sum_{m_3} g'_{\mathrm{f}1m_3} R_{12,03}|^2$ to a free proton-proton cross section is accomplished by the following formula:

$$\frac{\mathrm{d}\sigma_{\mathrm{fr}}^{\mathbf{P}}}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2} = \frac{2\pi}{\hbar}\frac{1}{v_{03}}\frac{\overline{\sum_{\mathrm{f}}|\sum_{m_3} g'_{\mathrm{f}1m_3} R_{12,03}|^2}}{\sum_{\mathrm{f}}\sum_{m_3}|g'_{\mathrm{f}1m_3}|^2}\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3) \cdot \delta(E_1+E_2-E_0-E_2); \tag{a}$$

$\mathrm{d}\sigma_{\mathrm{fr}}^{\mathbf{P}}/\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2$ is the cross section for free proton-proton scattering corresponding to the nuclear case and is therefore taken in a coordinate system in which the momenta of the colliding protons with relative velocity v_{03} are $\hbar\mathbf{k}_0$ and $\hbar\mathbf{k}_3$ and the polarization vector of the proton with index 3 is equal to $\mathbf{P}$. As in I, a sufficiently smooth dependence of $R_{12,03}$ on the momenta has been assumed, because the values for the momenta in the free scattering are somewhat different from those in the nuclear problem, as the energy conservation is not the same for both cases. This assumption may be only roughly correct for scattering angles very different from 45 degrees, in which case the nuclear proton has a high momentum and the two energy conservations become rather different.

$\mathrm{d}\sigma_{\mathrm{fr}}^{\mathbf{P}}/\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2$ is not a directly experimentally known quantity, and therefore it will be expressed in terms of the centre of mass cross section $\mathrm{d}\sigma_{\mathrm{fr}}^{\mathbf{P}}/\mathrm{d}\Omega'$. In the following, all primed quantities refer to the centre of mass system, which has the velocity v in the system used hitherto. We shall only consider the plane symmetric case, for which the index $\mathbf{P}$ may be omitted.

The Lorentz-transformation between the systems results in the following relations:

$$\frac{\mathrm{d}\sigma_{\mathrm{fr}}}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2} = \frac{\mathrm{d}k'_{1z}}{\mathrm{d}k_{1z}}\frac{\mathrm{d}k'_{2z}}{\mathrm{d}k_{2z}}\frac{\mathrm{d}\sigma_{\mathrm{fr}}}{\mathrm{d}\mathbf{k}'_1\mathrm{d}\mathbf{k}'_2} = \left(1-\frac{v^2}{c^2}\right)^{-1}\left(1-\frac{vp_1\cos\theta}{E_1}\right)^2\frac{\mathrm{d}\sigma_{\mathrm{fr}}}{\mathrm{d}\mathbf{k}'_1\mathrm{d}\mathbf{k}'}, \tag{b}$$

θ being the angle between the directions of an outgoing proton and the bombarding proton.

The following formula, connecting $\mathrm{d}\sigma_{\mathrm{fr}}/\mathrm{d}\mathbf{k}'_1\mathrm{d}\mathbf{k}'_2$ with the cross section $\mathrm{d}\sigma_{\mathrm{fr}}/\mathrm{d}\Omega'$, can be verified by integrating both sides over $\mathbf{k}'_2$ and $|\mathbf{k}'_1|$:

$$\frac{\mathrm{d}\sigma_{\mathrm{fr}}}{\mathrm{d}\mathbf{k}'_1\mathrm{d}\mathbf{k}'_2} = \frac{c^2\left(\dfrac{\mathbf{p}_1'^2}{E'_1}-\dfrac{\mathbf{p}'_1\cdot\mathbf{p}'_2}{E'^2}\right)}{|\mathbf{k}'_1|^3}\frac{\mathrm{d}\sigma_{\mathrm{fr}}}{\mathrm{d}\Omega'}\,\delta(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}'_0-\mathbf{k}'_3)\,\delta(E'_1+E'_2-E'_0-E'_3).$$

Because $\mathbf{p}'_2 = -\mathbf{p}'_1$ one may write

$$\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\mathbf{k}'_1\mathrm{d}\mathbf{k}'_2} = \frac{2c^2\hbar^3}{p'_1E'_1}\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\Omega'}\delta(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}'_0-\mathbf{k}'_3)\delta(E'_1+E'_2-E'_0-E'_3)$$
$$= \frac{2c^2\hbar^3}{p'_1E'_1}\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\Omega'}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3)\delta(E_1+E_2-E_0-E_3). \qquad \text{(c)}$$

Inserting (c) in (b),

$$\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2} = \left(1-\frac{v^2}{c^2}\right)^{-1}\left(1-\frac{vp_1\cos\theta}{E_1}\right)^2\frac{2c^2\hbar^3}{p'_1E'_1}\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\Omega'} \qquad \text{(d)}$$
$$\cdot\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3)\delta(E_1+E_2-E_0-E_3).$$

From the Lorentz transformation to the centre of mass system, in which the scattered protons, because of the symmetry, move orthogonally to their original directions, it follows that

$$v = \frac{c^2p_1\cos\theta}{E'_1}, \qquad p'_1 = p_1\sin\theta, \qquad E'_1 = \sqrt{p_1^2c^2\sin^2\theta+m^2c^4}.$$

Equation (d) now becomes

$$\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2} = \left(1-\frac{c^2p_1^2\cos^2\theta}{E_1^2}\right)\frac{2c^2\hbar^3}{p_1\sin\theta\sqrt{p_1^2c^2\sin^2\theta+m^2c^4}}\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\Omega'} \qquad \text{(e)}$$
$$\cdot\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3)\delta(E_1+E_2-E_0-E_3).$$

From (e) and (a) $\sum_t\overline{|\sum_{m_3}g'_{tlm_3}R_{12,03}|^2}$ may be calculated and inserted in eq. (1) of the text, giving the result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2} = \frac{v_{03}}{v_0}\left(1-\frac{c^2p_1^2\cos^2\theta}{E_1^2}\right)\frac{2c^2\hbar^3}{p_1\sin\theta\sqrt{p_1^2c^2\sin^2\theta+m^2c^4}}$$
$$\cdot\,\frac{\mathrm{d}\sigma_{\mathrm{tr}}}{\mathrm{d}\Omega'}\sum_{\mathrm{f}}\sum_m\overline{|g'_{tlm}|^2}\delta(E_1+E_2+E_n+E_b-E_3).$$

With

$$v_0 = c\sqrt{1-\frac{m^2c^4}{E_0^2}}, \qquad v_{03} = \sqrt{1-\left(\frac{m^2c^4}{2p_1^2c^2\sin^2\theta+m^2c^4}\right)^2}$$

and using the relation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1\mathrm{d}E_1\mathrm{d}\Omega_2\mathrm{d}E_2} = \frac{k_1k_2E_1E_2}{(\hbar c)^4}\frac{\mathrm{d}\sigma}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2} = \frac{k_1^2E_1^2}{(\hbar c)^4}\frac{\mathrm{d}\sigma}{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}$$

the expression (6) of the text is obtained.

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ON THE MASSES OF ELEMENTARY PARTICLES

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Abstract: Some speculations about self-masses of elementary particles are presented.

One of the apparent surprises in field theory is the insensitiveness of the axial-vector coupling constant renormalization in β-decay so far as radiative corrections from strong interactions are concerned. The argument is as follows. Empirically the vector coupling constants $g^{\mu}{}_{V}$ in μ-decay and $g^{\beta}{}_{V}$ in β-decay are equal [1]). Even with the assumption of a universal Fermi interaction this is surprising, for $g^{\beta}{}_{V}$ should have renormalization corrections from strong interactions. The most plausible explanation is the suggestion of Gershtejn and Zel'dovich [2]) and Feynman and Gell-Mann [1]), that the vector parts of μ and β-decay interactions satisfy a conservation relation. No such suggestion is possible for the axial-vector interaction, and $g^{\mu}{}_{A}$ and $g^{\beta}{}_{A}$ are indeed empirically different. Since experimentally $g^{\mu}{}_{A} \approx g^{\mu}{}_{V}$ for μ-decay, it is an attractive hypothesis to assume the same is true in the case of β-decay for the bare coupling constants and to ascribe the deviation of the observed ratio $|g^{\beta}{}_{A}/g^{\beta}{}_{V}|$ from unity to renormalization effects. There still remains the quantitative mystery that this deviation ($-g^{\beta}{}_{A}/g^{\beta}{}_{V} = 1.25 \pm 0.04$) is not as large as one may reasonably expect from the very strong interactions which produce the radiative corrections.

One wonders whether this insensitivity to higher order corrections is not a general feature of all self-charge renormalization effects, especially as a similar situation seems to prevail in the case of self-masses also. One of the strongly suggestive arguments in favour of this is the remark due to Gell-Mann and to Schwinger [3]), who show that the relation between baryon masses, experimentally accurate to 5 %,

$$2(m_N + m_\Xi) = (m_\Lambda + 3m_\Sigma),$$

can be deduced by applying *second order* perturbation theory to K-meson interactions, assuming that π-mesons interact similarly with all hyperons (global symmetry).

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It may be conjectured from these examples that for the computation of self-charge and self-mass effects use of second order perturbation theory may not be too misleading. On this basis we wish to point out that one may also hope to understand the remarkable empirical relation [4])

$$\frac{m_p}{m_e} \approx \frac{g_\pi^2}{e^2}. \tag{1}$$

With the present value of $g_\pi^2 \approx 15$, this is accurate to nearly 10 %. Using second order perturbation theory, the second order contribution to a fermion self-mass when it interacts with a boson equals

$$\delta m = A g_0^2 \frac{m_0}{4\pi} \log \frac{X}{m_0}. \tag{2}$$

Here m_0 is the bare mass, X some cut-off mass, g_0 the bare coupling constant ($e^2/4\pi = 1/137$) in the units used) and the constant A depends on the spin and parity of the boson:

$$\begin{aligned} A &= +6 \quad \text{for vector bosons,} \\ &= +1 \quad \text{for pseudoscalar bosons,} \\ &= -3 \quad \text{for scalar bosons.} \end{aligned}$$

The physical mass m is given by

$$m = m_0 + \delta m. \tag{3}$$

Using relations (2), (3) and assuming that

(1) a nucleon and an electron have the same bare mass m_0,
(2) m_0 is so small that it can be neglected in comparison with δm,
(3) to this order $g_0 \approx g$, where g is the renormalized constant,
(4) the same cut-off mass X should always be used,

we have the result

$$\frac{m_p}{m_e} = \frac{3g_{\pi N}^2 + g_{K\Lambda}^2 + 3g_{K\Sigma}^2}{6e^2} \tag{4}$$

to be compared with (1). For this argument it is imperative that the π-meson be pseudoscalar; for with a scalar meson one would get a negative self-mass and the bare mass could never be considered small. (This may be an argument in favour of K-mesons also being pseudoscalar relative to the ΛN and ΣN systems.)

The same type of reasoning could be carried through for other hyperons. The Nambu-Fröhlich [5]) observation that the masses of all elementary particles are integer or half-integer multiples of 137 appears here in the form of a (somewhat mystical) statement that not merely $1/e^2$, but also g_π^2, $g_{K\Lambda}^2$

etc. are nearly integers. From

$$\frac{m_\mu}{m_e} \approx 137 \times \frac{3}{2}$$

one may infer that the μ-meson must possess some strong interaction with a coupling constant of the order of unity.

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SCHEMA EINER QUANTENMECHANIK MIT INDEFINITER METRIK

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Abstract: The problem is examined how to change the axioms of quantum theory in order to introduce an indefinite metric. There are several inequivalent ways to divide the total Hilbert space of indefinite metric into two parts, one of which is of positive definite, the other of negative definite metric (so-called "physical" and "ghost-states").

In view of the arbitrariness in the choice of such decompositions the question is raised whether the completeness of the quantal description only requires the knowledge of the state-vectors and the hermitian operators belonging to the observables, or whether it is necessary to have further knowledge in order to privilege one mode of decomposition of the space into a positive and a negative part.

In the paper, the second suggestion is denied. Every commuting system of hermitian operators points to a definite decomposition in a *natural* way, provided it is "large enough". We call "decomposing" every system of commuting operators which divides (in a certain sense) the total Hilbert space into a positive and a negative part. Not only does a decomposing system induce the decomposition, but there is associated with it a new positive definite metric of the *total* space. In this way a probabilistic interpretation is valid and the usual axiomatics is applicable. With respect to decomposing systems which induce the same decomposition, all results belonging to a quantum theory with positive definite metric are conserved.

But there exists another case: the hermitian operator A may belong simultaneously to two decomposing systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ and the decomposition relative to $\mathfrak{S}_1$ may not agree with the decomposition relative to $\mathfrak{S}_2$. Then, when analysing a quantum state with the help of observables represented by operators belonging to $\mathfrak{S}_1$, we may find an expectation value $\bar{A}^{(1)}$ *different* from the expectation value $\bar{A}^{(2)}$ of A with respect to an experiment that determines (in principle) the $\mathfrak{S}_2$-observables.

1. Einführung

Die folgenden Ausführungen wurden durch eine Arbeit von Ascoli und Minardi [2]) angeregt. In dieser Arbeit wird behauptet, daß sich jede Quantentheorie mit indefiniter Metrik auf eine solche mit positiv-definiter Metrik zurückführen lasse, so daß die Einführung einer nicht-definiten Metrik zwar ein unter Umständen eleganter und nützlicher, das Wesen der Sache aber nicht verändernder Kunstgriff ist.

Es soll jedoch das Schema einer Quantentheorie skizziert werden, das sich beim Vorliegen indefiniter Metrik etwas vom Dirac-schen Schema [4]) unterscheidet, wodurch ein qualitativ neuer Zug auftritt: Ist A eine Observable, dann *kann* (aber muß nicht!) der Fall eintreten, daß der Erwartungs-

wert $\bar{A}$ von A nicht bestimmt ist, wenn lediglich der Zustandsvektor ψ gegeben ist. Es kommt vielmehr beim Vorliegen indefiniter Metrik vor, daß $\bar{A}$ erst dann bestimmt ist, wenn Näheres über die Art und Weise bekannt ist, wie dieser Erwartungswert experimentell bestimmt wird. Werde z.B. $\bar{A}$ durch Experimente bestimmt, bei denen gleichzeitig mit A die Observablen $A_1, \ldots, A_r$ meßbar sind. Dann ist durch das Vorgeben des Zustandsvektors ψ und durch die Angabe, daß mit A auch gleichzeitig $A_1, \ldots, A_r$ meßbar sind, der Erwartungswert $\bar{A}$ von A eindeutig bestimmt. Der Erwartungswert von A bezüglich der Messung von A gleichzeitig mit einem anderen System $A'_1, \ldots, A'_s$ (das untereinander und mit A kommutiert), braucht nicht mit $\bar{A}$ übereinzustimmen.

Die Dirac-sche „bra-ket-Schreibweise" findet in der vorliegenden Arbeit keine Anwendung, da wir mit mehreren Metriken gleichzeitig rechnen müssen. Es werden die Zustandsvektoren (für einen gegebenen Zeitpunkt) mit $\psi, \psi_1, \ldots$ usw. bezeichnet. Die Zustandsvektoren bilden einen linearen komplexen Raum, auf dem Metriken (Skalarprodukte, Bilinearformen) $\langle\psi_1, \psi_2\rangle$, $\langle\psi_1, \psi_2\rangle_r$ usw. erklärt sind bzw. werden. Zu jeder solchen Metrik gehört eine bestimmte Norm $\langle\psi, \psi\rangle$, $\langle\psi, \psi\rangle_r$ usw. Die Begriffe „unitär" und „Hermitesch" sind nicht absolut, sondern bezüglich einer bestimmten Metrik gegeben.

2. Der Raum der Zustandsvektoren

Es sei $\mathfrak{H}$ ein komplexer Hilbert-Raum mit einer indefiniten Metrik

$$\langle\psi_1, \psi_2\rangle; \qquad \psi_1, \psi_2 \in \mathfrak{H}.$$

Von seinen Vektoren nehmen wir an, daß sie die möglichen Zustände einer quantentheoretischen Gegebenheit beschreiben. Von der gegebenen Metrik wollen wir nur annehmen, daß sie nicht entartet ist, d.h. ist $\psi \in \mathfrak{H}$ ein beliebiger Vektor, so existiert mindestens ein zweiter Vektor $\psi' \in \mathfrak{H}$ mit $\langle\psi, \psi'\rangle \neq 0$. (Ist die Metrik entartet, so gibt es eine eindeutige Zerlegung von $\mathfrak{H}$ in zwei zueinander orthogonale Teilräume, von denen der eine eine nichtausgeartete Metrik trägt, während auf dem anderen die Metrik identisch verschwindet. Wir beschränken uns dann auf den ersteren, den zweiten lassen wir weg.) Aus der Formel

$$\langle\lambda\psi_1+\psi_2, \lambda\psi_1+\psi_2\rangle = |\lambda|^2\langle\psi_1, \psi_1\rangle+\langle\psi_2, \psi_2\rangle+\lambda\langle\psi_1, \psi_2\rangle+\lambda^*\langle\psi_2, \psi_1\rangle \quad (1)$$

folgt, daß es Vektoren in $\mathfrak{H}$ gibt, deren Norm von Null verschieden ist; denn sonst wäre die Metrik identisch Null.

Vor dem Aufbau eines quantentheoretischen Schemas müssen wir noch das Orthonormierungsverfahren für einen Hilbert-Raum mit indefiniter, nicht ausgearteter Metrik besprechen.

3. Das Orthonormierungsverfahren

Eine Menge Ω von Vektoren aus $\mathfrak{H}$ wollen wir ein *Orthonormalsystem* nennen, wenn gilt:

$$\begin{aligned} &\langle \psi, \psi \rangle = \pm 1 \quad \text{für alle} \quad \psi \in \Omega, \\ &\langle \psi_1, \psi_2 \rangle = 0 \quad \text{für alle} \quad \psi_1, \psi_2 \in \Omega \quad \text{mit} \quad \psi_1 \neq \psi_2. \end{aligned} \tag{2}$$

Jede orthonormale Menge zerfällt in zwei getrennte Teile Ω_+ und Ω_-: Gerade dann gehört ein Vektor ψ aus Ω zu Ω_+, wenn die Norm von ψ positiv ist. Analoges gilt für die Bestimmung von Ω_-.

Wir bezeichnen ferner mit

$$(\Omega) \quad \text{bzw.} \quad (\Omega_+) \quad \text{bzw.} \quad (\Omega_-)$$

die Gesamtheit der Vektoren

$$\sum \alpha_k \psi_k, \qquad \sum |\alpha_k|^2 < \infty$$

mit

$$\psi_k \in \Omega \quad \text{bzw.} \quad \psi_k \in \Omega_+ \quad \text{bzw.} \quad \psi_k \in \Omega_-.$$

Da die Metrik auf (Ω_+) und (Ω_-) definit ist, sind $(\Omega_\pm)$ eigentliche Hilbert-Räume. (Einen Hilbert-Raum mit definiter Metrik bezeichnen wir als „eigentlich".) Es ist klar, daß $\langle \psi_+, \psi_- \rangle = 0$ für beliebige $\psi_+ \in (\Omega_+)$, $\psi_- \in (\Omega_-)$ ist.

Wir fügen nun zu den Eigenschaften von $\mathfrak{H}$ ergänzend hinzu: Ist Ω ein Orthonormalsystem, so sind alle Vektoren aus (Ω) in $\mathfrak{H}$ enthalten (Vollständigkeit von $\mathfrak{H}$).

Hilfssatz: Ist Ω ein Orthonormalsystem und $\psi \in \mathfrak{H}$, so gibt es ein weiteres Orthonormalsystem Ω' mit

$$\psi \in (\Omega') \quad \text{und} \quad \Omega \subset \Omega'.$$

Beweis: Es sind

$$\langle \psi, \psi_+ \rangle, \quad \psi_+ \in (\Omega_+); \qquad \langle \psi, \psi_- \rangle, \quad \psi_- \in (\Omega_-)$$

zwei stetige Linearformen auf dem eigentlichen Hilbert-Räumen $(\Omega_\pm)$. Es gibt folglich Vektoren $\psi_1 \in (\Omega_+)$, $\psi_2 \in (\Omega_-)$ mit

$$\begin{aligned} &\langle \psi, \psi_+ \rangle = \langle \psi_1, \psi_+ \rangle \quad \text{für alle} \quad \psi_+ \in (\Omega_+), \\ &\langle \psi, \psi_- \rangle = \langle \psi_2, \psi_- \rangle \quad \text{für alle} \quad \psi_- \in (\Omega_-). \end{aligned}$$

Wir bilden $\psi_3 = \psi - \psi_1 - \psi_2$ und finden, daß

$$\begin{aligned} &\langle \psi_3, \psi_+ \rangle = 0 \quad \text{für alle} \quad \psi_+ \in (\Omega_+), \\ &\langle \psi_3, \psi_- \rangle = 0 \quad \text{für alle} \quad \psi_- \in (\Omega_-) \end{aligned}$$

ist. Es ist folglich ψ_3 orthogonal zu (Ω) und deshalb erst recht zu Ω. Ist nun

$\langle \psi_3, \psi_3 \rangle \neq 0$, so steht der Normierung nichts mehr im Wege und Ω' besteht aus Ω und dem Element

$$\frac{\sqrt{|\langle \psi_3, \psi_3 \rangle|}}{\langle \psi_3, \psi_3 \rangle} \psi_3 .$$

Besondere Überlegungen erfordert der Fall $\langle \psi_3, \psi_3 \rangle = 0$. Es gibt dann einen Vektor ψ_4 in $\mathfrak{H}$ mit $\langle \psi_3, \psi_4 \rangle \neq 0$; denn gemäß unserer Voraussetzung ist die Metrik nicht entartet. Nach Obigem existiert für diesen Vektor eine Zerlegung $\psi_4 = \psi'_4 + \psi_5$ mit $\psi_5 \in (\Omega)$ und mit zu (Ω) orthogonalem ψ_5. Da $\langle \psi_3, \psi_5 \rangle = 0$ ist — denn ψ_3 ist Orthogonal zu (Ω) — ist $\langle \psi_3, \psi'_4 \rangle \neq 0$. Wir können also gleich annehmen, daß ψ_4 zusätzlich noch orthogonal zu (Ω) ist. Wir wählen nun ein λ so, daß

$$\begin{aligned} |\lambda|^2 \langle \psi_4, \psi_4 \rangle + \lambda \langle \psi_4, \psi_3 \rangle + \lambda^* \langle \psi_3, \psi_4 \rangle &\neq 0 \\ |\lambda|^2 \langle \psi_4, \psi_4 \rangle - \lambda \langle \psi_4, \psi_3 \rangle - \lambda^* \langle \psi_3, \psi_4 \rangle &\neq 0 \\ |\lambda|^2 \langle \psi_4, \psi_4 \rangle + \lambda \langle \psi_4, \psi_3 \rangle - \lambda^* \langle \psi_3, \psi_4 \rangle &= 0 \end{aligned}$$

ist, was wegen $\langle \psi_3, \psi_4 \rangle \neq 0$ geht. Ein Blick auf Formel (1) lehrt, daß mit dem so bestimmten λ wegen $\langle \psi_3, \psi_3 \rangle = 0$

$$\begin{gathered} \langle \psi_3 + \lambda \psi_4, \psi_3 + \lambda \psi_4 \rangle \neq 0, \qquad \langle \psi_3 - \lambda \psi_4, \psi_3 - \lambda \psi_4 \rangle \neq 0, \\ \langle \psi_3 + \lambda \psi_4, \psi_3 - \lambda \psi_4 \rangle = 0 \end{gathered} \tag{x}$$

ist. Da ψ_3 und ψ_4 orthogonal zu (Ω) ist, sind es auch die Elemente

$$\psi_6 = \psi_3 + \lambda \psi_4, \qquad \psi_7 = \psi_3 - \lambda \psi_4$$

und wegen (x) sind sie auch untereinander orthogonal. Da sie schließlich beide nicht-verschwindende Norm besitzen, können wir sie auf ± 1 normieren. Sie bilden dann zusammen mit Ω das gesuchte Orthonormalsystem Ω'; denn

$$\psi = \tfrac{1}{2}\psi_7 + \tfrac{1}{2}\psi_6 + \psi_1 + \psi_2 .$$

Damit haben wir das für die üblichen Schlüsse Wesentliche [1]) geleistet und können als gesichert ansehen:

Satz 1: Ist Ω^0 irgendein Orthonormalsystem von $\mathfrak{H}$, so gibt es ein Orthonormalsystem Ω von $\mathfrak{H}$ mit

a) $(\Omega) = \mathfrak{H}$ und
b) $\Omega^0 \subset \Omega$.

Ein Orthonormalsystem mit der Eigenschaft a) heißt vollständig.

Wir können nun leicht die Zerlegungen von $\mathfrak{H}$ in Teile mit positiver und negativer Norm (sog. „physikalische" und „Geisterzustände") beschreiben.

4. Zerlegung von $\mathfrak{H}$ in zwei eigentliche Hilbert-Räume

Ist Ω ein vollständiges Orthonormalsystem, so ist

$$\mathfrak{H} = (\Omega_+) + (\Omega_-) \tag{3}$$

eine Zerlegung von $\mathfrak{H}$ in eine direkte Summe und die Metrik ist auf (Ω_+) positiv und auf (Ω_-) negativ definit. Mit anderen Worten, $(\Omega_\pm)$ sind eigentliche Hilbert-Räume mit positiver bzw. negativer Metrik.

Man ist nun bestrebt, die physikalischen Zustände durch Vektoren aus (Ω_+) darzustellen und den Vektoren aus (Ω_-) keine (zumindest keine unmittelbare) physikalische Bedeutung beizumessen [2,3]. Hieraus schließen Ascoli und Minardi [2], daß man jede physikalische Theorie auch in (Ω_+) allein — wenn auch eventuell unbequemer — formulieren könne. Diese Verfasser haben dabei offenbar übersehen, daß die Zerlegung (3) nicht eindeutig ist. In der Tat, ist Ω' ein beliebiges anderes Orthonormalsystem, so ist

$$\mathfrak{H} = (\Omega'_+)+(\Omega'_-) \tag{3a}$$

unter Umständen eine völlig andere Zerlegung von $\mathfrak{H}$ und die Transformation

$$(\Omega_+) \to (\Omega'_+)$$

ist nur unitär (d.h. normerhaltend), wenn $(\Omega_+) = (\Omega'_+)$, die Zerlegungen also identisch sind. Mit Hilfe von Satz 1 kann man aber beliebig viele verschiedene Zerlegungen von $\mathfrak{H}$ aufbauen. Ist z.B. ψ_+ ein Vektor aus (Ω_+), ψ_- einer aus (Ω_-), so hat der Vektor $\psi' = 2\psi_+ - \psi_-$ die Norm 1 und folglich gibt es ein vollständiges Orthonormalsystem Ω' mit $\psi' \in \Omega'_+$.

Liefert also eine Theorie einen Hilbert-Raum mit indefiniter Metrik, so liegt in der Zuordnung der „physikalischen" Zustände zu den Vektoren aus $\mathfrak{H}$ der Zerlegung (3) ein Akt völliger Willkür: Es wird willkürlich eine Zerlegung (3) ausgewählt, wobei die verschiedenen Möglichkeiten der Auswahl dieser Zerlegung zu in $\mathfrak{H}$ *inäquivalenten* eigentlichen Hilbert-Räumen führen. Eine willkürfreie Zerlegung in „physikalische' und „Geisterzustände" ist deshalb nicht ohne weiteres möglich. Wir werden zeigen, daß trotz dieses Umstandes eine willkürfreie und sinnvolle Quantentheorie mit indefiniter Metrik möglich ist.

5. Zerlegende Systeme Hermitescher Operatoren

Sei A ein Hermitescher Operator, d.h. es sei

$$\langle A\psi_1, \psi_2\rangle = \langle \psi_1, A\psi_2\rangle, \qquad \psi_1, \psi_2 \in \mathfrak{H}.$$

Es ist trivial, daß dann

$$\langle \psi, A\psi\rangle = \langle A\psi, \psi\rangle (= \langle \psi, A\psi\rangle^*)$$

immer reell ist. Die Hermiteschen Operatoren haben also reelle „Erwartungswerte" und es ist die Beziehung

Observable $\leftrightarrow$ Hermitescher Operator gerettet, wenn es gelingt, eine entsprechende Normierungsvorschrift zu gewinnen.

Dies geht tatsächlich. Der eigentliche Trick dabei ist: Die Normierungs-

vorschrift ist nicht für alle Hermiteschen Operatoren einheitlich gegeben, sondern jedem „genügend vollständigem", kommutierenden (d.h. gleichzeitig meßbaren!) System von Hermiteschen Operatoren entspricht eine diesem System eigentümliche Normierungsvorschrift.

Hat man einen Hermiteschen Operator A mit reinem Punktspektrum (was wir der Einfachheit halber immer annehmen wollen), so kann man ein Orthonormalsystem Ω bestimmen, dessen Elemente Eigenfunktionen von A sind; denn erstens sind die zu verschiedenen Eigenwerten gehörenden Vektoren orthogonal zueinander und zweitens kann man den zu einem bestimmten Eigenwert λ gehörenden Raum der Eigenvektoren nach der in Nr. 3 beschriebenen Methode orthonormalisieren. Bei einem Hermiteschen Operator kann es nun vorkommen, daß es zu gleichem Eigenwert λ sowohl Eigenfunktionen in Ω_+ als auch in Ω_- gibt. Zu einem solchen Operator kann man keine eindeutige Zerlegung (3) konstruieren. Analoges gilt für Systeme kommutierender Hermitescher Operatoren.

Definition: Ein System

$$\mathfrak{S} = \{A_1, A_2, A_3, \ldots, A_s\}$$

von kommutierenden Hermiteschen Operatoren heißt genau dann *zerlegend* (oder einfach „z-System"), wenn gilt: Ist λ_k ein Eigenwert von $A_k (k = 1, \ldots, s)$, so ist der lineare Raum aller ψ mit

$$A_k \psi = \lambda_k \psi \qquad (k = 1, \ldots, s)$$

entweder leer oder die Metrik ist auf ihm definit. Nach der obigen Bemerkung über Orthonormalsysteme, die aus Eigenvektoren bestehen, gilt dann

Satz 2: Zu jedem zerlegenden System

$$\mathfrak{S} = \{A_1, A_2, \ldots, A_s\}$$

kommutierender Hermitescher Operatoren gehört genau eine Zerlegung

$$\mathfrak{Z} = \mathfrak{Z}_+(\mathfrak{S}) + \mathfrak{Z}_-(\mathfrak{S})$$

mit

$$A_k \mathfrak{Z}_+(\mathfrak{S}) \subseteq \mathfrak{Z}_+(\mathfrak{S}), \qquad A_k \mathfrak{Z}_-(\mathfrak{S}) \subseteq \mathfrak{Z}_-(\mathfrak{S}),$$

und diese Zerlegung ist wiefolgt gewinnbar: Ist Ω ein vollständiges Orthonormalsystem, das aus gemeinsamen Eigenfunktionen der Operatoren A_k besteht, so ist

$$\mathfrak{Z}_+(\mathfrak{S}) = (\Omega_+), \qquad \mathfrak{Z}_-(\mathfrak{S}) = (\Omega_-).$$

Ganz einfach ergibt sich heraus:

Satz 3: Seien $\mathfrak{S}^0$ und $\mathfrak{S}$ zwei Systeme kommutierender Hermitescher Operatoren und jeder zu $\mathfrak{S}^0$ gehörende Operator gehöre auch zu $\mathfrak{S}$: $\mathfrak{S}^0 \subset \mathfrak{S}$. Ist dann $\mathfrak{S}^0$ zerlegend, so ist es auch $\mathfrak{S}$ und es gilt

$$\mathfrak{Z}_+(\mathfrak{S}) = \mathfrak{Z}_+(\mathfrak{S}^0), \qquad \mathfrak{Z}_-(\mathfrak{S}) = \mathfrak{Z}_-(\mathfrak{S}^0).$$

Denn jedes zur Konstruktion von $\mathfrak{Z}_+(\mathfrak{S}^0)$ usw. benutzbare Orthonormalsystem gemeinsamer Eigenfunktionen der Operatoren von $\mathfrak{S}$ ist auch zur Konstruktion von $\mathfrak{Z}_-(\mathfrak{S})$ usw. verwendbar.

Bemerkung: Ist die Metrik definit, so ist *jedes* System zerlegend und $\mathfrak{Z}_+(\mathfrak{S}) = \mathfrak{Z}$, $\mathfrak{Z}_-(\mathfrak{S}) = 0$. Hiernach sind alle Größen, die abhängen von 1) Zustandsvektoren, 2) einem Hermiteschen Operator und 3) einem z-System von Operatoren, das diesen enthält, so beschaffen, daß die Abhängigkeit von z-System bei positiv-definiter Metrik unterbleibt. Diese Abhängigkeit ist also ein Charakteristikum der indefiniten Metrik allein!

Wir können jetzt eine Quantentheorie wiefolgt aufbauen: Bleibt man innerhalb eines z-Systems (oder eines dieses System umfassenden Systems), so geschieht alles wie in einer Theorie mit positiv-definiter Metrik. Der Übergang zu einem anderen System gleichzeitig meßbarer Größen erfolgt jedoch (unter Umständen) unter gleichzeitiger Abänderung der Norm. Im Folgenden skizzieren wir wichtige Züge eines solchen Aufbaues.

6. Norm, Erwartungswert und Übergangswahrscheinlichkeit

Wir nehmen an, $\mathfrak{S} = \{A_1, A_2, A_3, \ldots\}$ sei ein zerlegendes System untereinander kommutierender Hermitescher Operatoren.

a) Wir definieren eine $\mathfrak{S}$-*Metrik in* $\mathfrak{Z}$ wie folgt: Sind ψ, ψ' zwei Vektoren aus $\mathfrak{Z}$, so gibt es genau eine Zerlegung

$$\begin{aligned} \psi &= \psi_+ + \psi_- \\ \psi' &= \psi'_+ + \psi'_- \end{aligned} \quad \text{mit} \quad \begin{aligned} &\psi_+, \psi'_+ \in \mathfrak{Z}_+(\mathfrak{S}), \\ &\psi_-, \psi'_- \in \mathfrak{Z}_-(\mathfrak{S}). \end{aligned} \tag{4a}$$

Wir setzen dann

$$\langle \psi, \psi' \rangle_{\mathfrak{S}} = \langle \psi_+, \psi'_+ \rangle - \langle \psi_-, \psi'_- \rangle. \tag{4b}$$

Dies ist eine durch $\mathfrak{S}$ *eindeutig bestimmte, positiv definite Metrik.*

b) Alle Operatoren A_k aus $\mathfrak{S}$ sind Hermitesch bezüglich der eben bestimmten Metrik $\langle \psi, \psi' \rangle_{\mathfrak{S}}$. Dies folgt aus zwei Tatsachen: Einmal sind die A_k Hermitesch bezüglich der indefiniten Metrik $\langle \psi, \psi' \rangle$ nach Voraussetzung und zum anderen lassen sie nach Satz 2 die zur Definition der positiv definiten Metrik benutzten Teilräume $\mathfrak{Z}_+(\mathfrak{S})$, $\mathfrak{Z}_-(\mathfrak{S})$ invariant.

c) Als Erwartungswert für den zu $\mathfrak{S}$ gehörenden Operator A definieren wir

$$\bar{A} = \frac{\langle \psi, A\psi \rangle_{\mathfrak{S}}}{\langle \psi, \psi \rangle_{\mathfrak{S}}}. \tag{5}$$

d) Als Übergangswahrscheinlichkeit von Zustand ψ_1 zu Zustand ψ_2 bezüglich der Messung aller $A_k \in \mathfrak{S}$ definieren wir

$$W_{\psi_1, \psi_2}(\mathfrak{S}) = \frac{|\langle \psi_1, \psi_2 \rangle_{\mathfrak{S}}|^2}{\langle \psi_1, \psi_1 \rangle_{\mathfrak{S}} \langle \psi_2, \psi_2 \rangle_{\mathfrak{S}}}. \tag{6}$$

Die Messung der Übergangswahrscheinlichkeit besteht in einer „Spektralzerlegung" der Zustände ψ_1 und ψ_2 nach einem einzigen vollständigem System gemeinsam meßbarer Observablen. Es ist deshalb prinzipiell angänglich, daß in die Definition der Übergangswahrscheinlichkeit die zu ihrer Messung benutzten Operatoren eingehen.

Folgerung 1: Wegen Formel (6) stellen wir fest, daß stets

$$W_{\psi,\psi'} \geqq 0 \tag{7}$$

ist.

Folgerung 2: Da die Räume $\mathfrak{Z}_+(\mathfrak{S})$, $\mathfrak{Z}_-(\mathfrak{S})$ zueinander orthogonal sind (und zwar sowohl bezüglich der Metrik $\langle\psi, \psi'\rangle$ als auch bezüglich der Metrik $\langle\psi, \psi'\rangle_{\mathfrak{S}}$), ist automatisch

$$W_{\psi_+,\psi_-}(\mathfrak{S}) = 0 \quad \text{für} \quad \psi_+ \in \mathfrak{Z}_+(\mathfrak{S}), \qquad \psi_- \in \mathfrak{Z}_-(\mathfrak{S}). \tag{8}$$

Dies garantiert, daß bezüglich der Analyse der Zustände nach Operatoren aus $\mathfrak{S}$ Übergänge von Zuständen aus $\mathfrak{Z}_+(\mathfrak{S})$ in solche aus $\mathfrak{Z}_-(\mathfrak{S})$ und umgekehrt absolut verboten sind.

Folgerung 3: Sei A aus $\mathfrak{S}$ und sei ψ Eigenvektor von A mit Eigenwert λ. Es ist

$$\bar{A} = \lambda. \tag{9}$$

Zusammenfassend können wir sagen: Zu jedem Satz kommutierender Hermitescher Operatoren, die ein zerlegendes System bilden, gehört genau eine Zerlegung (3) und genau eine positiv-definite Metrik (4) auf $\mathfrak{Z}$. Bleibt man innerhalb dieses Systems kommutierender Operatoren, so bleiben alle Formeln und Schlüsse wie in einer Quantentheorie mit positiv-definiter Metrik. Übergänge von $\mathfrak{Z}_+(\mathfrak{S})$ nach $\mathfrak{Z}_-(\mathfrak{S})$ und umgekehrt sind dabei verboten.

Der Übergang zu einem *anderen* z-System kommutierender Operatoren entspricht im allgemeinen dem Übergang zu einer *anderen* Zerlegung von $\mathfrak{Z}$ und einer *anderen* definiten Metrik (4). Dieser Übergang ist *nicht* durch eine unitäre Transformation mit der ursprünglichen Zerlegung verbunden.

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ON THE CONNECTION BETWEEN THE CLUSTER MODEL AND THE SU_3 COUPLING SCHEME FOR PARTICLES IN A HARMONIC OSCILLATOR POTENTIAL

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Abstract: It is shown that the cluster model of Wildermuth and Kanellopoulos provides an alternative description of certain states in the SU_3 coupling scheme of Elliott.

In a recent series of papers [1,2]), Wildermuth and Kanellopoulos have discussed a "cluster model" for particles moving independently in a harmonic oscillator field. They utilize the separability of this field to write their wave functions in terms of clusters of particles moving relative to each other. They then argue that, when interactions are added, the states of lowest energy will be those in which the clusters remain unexcited, but are in different states of relative motion. This set of states is also found to have certain properties characteristic of a rotational band. For example, they consider Be^8 in terms of two unexcited α-particles in relative motion. Because of the exclusion principle, their relative oscillations must have at least four quanta, and so one obtains ^{11}S, ^{11}D and ^{11}G levels. Similarly, F^{19} is considered in terms of unexcited $O^{16}+H^3$; the lowest states of relative motion have in this case six quanta, giving rise to ^{22}S, ^{22}D, ^{22}G and ^{22}I levels.

For low-lying nuclear states, such as those just mentioned, the cluster model can be thought of as a coupling scheme for the particles outside closed shells. The problem arises of the relationship between this coupling scheme and other known coupling schemes appropriate to the harmonic oscillator potential. Connected with this problem is that of characterizing the interactions which are diagonal in the cluster scheme. It is the purpose of this note to show that the cluster model coupling scheme is intimately connected with the SU_3 coupling scheme recently introduced by Elliott [3]). The latter gives a classification of states which is similar to that obtained by considering the individual particles as moving independently in a spheroidal rotating field. In accordance with this physical picture, the SU_3 coupling scheme is realized for inter-particle forces of quadrupole type, which can alternatively be described in terms of a deformation of the field (cf. also ref.[4])).

Before considering these relationships, we note the connection of the

cluster model scheme with the supermultiplet classification of Wigner [5]). A cluster which is a closed shell in its own harmonic oscillator field is invariant under transformation by SU_4 in charge-spin space. A nucleus consisting entirely of closed-shell clusters is therefore also invariant, and this uniquely determines its supermultiplet classification. If one of the clusters is not a closed shell, the supermultiplet quantum numbers of the entire nucleus are determined by those of this cluster. The α-cluster states of Be^8 therefore have supermultiplet character [44], which uniquely specifies them. Since for the p-shell, the supermultiplet and SU_3 classifications are identical, the identity of the SU_3 and cluster wave functions for B^8 follows. The F^{19} cluster model states have supermultiplet character [44443]. This does not uniquely specify them, but, as we shall see, here again the cluster model and SU_3 coupling schemes are equivalent.

The SU_3 coupling scheme is based on the transformation properties of the wave functions under the group of 3×3 unimodular (i.e., of determinant unity) unitary matrices. For our purposes it is convenient to define the action of the elements of SU_3 in the following manner. Let C_{x_i}, C_{y_i}, C_{z_i} be creation operators for x, y and z quanta of the i^{th} particle. We get all the $\frac{1}{2}(N_i+1)(N_i+2)$ single-particle states with N_i quanta by forming the products

$$C_{x_i}^{n_{x_i}} C_{y_i}^{n_{y_i}} C_{z_i}^{n_{z_i}} \psi_0, \qquad n_{x_i}+n_{y_i}+n_{z_i} = N_i, \tag{1}$$

where ψ_0 signifies the 1s state of the oscillator. We now replace C_{x_i}, C_{y_i}, and C_{z_i} by their linear combinations defined by a 3×3 unimodular unitary matrix

$$\begin{matrix} C_{x_i} & & C'_{x_i} \\ C_{y_i} & \rightarrow & C'_{y_i} \\ C_{z_i} & & C'_{z_i} \end{matrix} = \left[SU_3\right] \begin{matrix} C_{x_i} \\ C_{y_i} \\ C_{z_i} \end{matrix}. \tag{2}$$

Since this is a linear transformation, each of the products of (1) will be transformed into a linear combination of products of (1). In this sense, the harmonic oscillator states of N_i quanta carry a $\frac{1}{2}(N_i+1)(N_i+2)$-dimensional representation of SU_3. This representation is known to be irreducible; it is labelled by the partition $[N_i 00]$. If we restrict ourselves to the subgroup of SU_3 consisting of 3-dimensional rotation matrices (R_3), this representation is reducible into D^{N_i}, D^{N_i-2}, ..., $D^{0 \text{ or } 1}$. This is the familiar rule giving the possible angular momenta of a particle with N_i oscillator quanta.

If we have a many-particle wave function

$$C_{x_1}^{n_{x_1}} C_{y_1}^{n_{y_1}} C_{z_1}^{n_{z_1}} \ldots C_{x_f}^{n_{x_f}} C_{y_f}^{n_{y_f}} C_{z_f}^{n_{z_f}} \psi_0, \tag{3}$$

we define SU_3 to act by performing the same linear transformation (2) on the creation operators of each particle. This will cause the many-particle states of a given permutation symmetry to transform amongst each other, carrying

a representation (in general reducible) of SU_3. Elliott has discussed a coupling scheme which utilizes those linear combinations of the many-particle functions (3) which carry irreducible representations of SU_3.

To make contact with the cluster model, we introduce a new set of coordinates

$$\varrho_i = \sum_{j=1}^{f} a_{ij} \mathbf{r}_j \quad (i = 1, \ldots, f), \tag{4}$$

where a_{ij} is a real orthogonal transformation. For example, some of the ϱ_i might describe internal degrees of freedom of clusters, and the others their relative coordinates, etc. The corresponding creation operators are simply

$$C_{\rho_i} = \sum_{j=1}^{f} a_{ij} C_{\mathbf{r}_j} \quad (i = 1, \ldots, f). \tag{5}$$

Therefore, if we perform the transformation (2) on the x, y and z components of all the $C_{\mathbf{r}_i}$, we induce the same transformation on the x, y and z components of the C_{ρ_i}. In particular, these transformations leave invariant the number of quanta associated with each of the new degrees of freedom, and merely redistribute them amongst the x, y and z directions. *Thus, the sets of states considered in the cluster model carry representations of* SU_3.

In simple cases, this representation will be irreducible. As an example, consider the cluster model states of F^{19} associated with six quanta (the lowest number) of relative motion of unexcited O^{16} and H^3. These are seen to transform under SU_3 in the same way as single-particle states with $N_i = 6$, and thus carry the irreducible representation [600] or (60) in Elliott's notation. Since there is only one set of states in the configuration $(1s)^4\ (1p)^{12}\ (2s\ 1d)^3$ having these transformation properties, the cluster model states must be identical with the particular set discussed by Elliott. From this follows a number of simple geometrical properties of the cluster states, associated with the rotational interpretation of the SU_3 scheme. It also follows that the cluster model is a good approximation for forces which can be described in terms of a spheroidal field.

In turn, the simple physical picture of relative oscillations of two unexcited clusters provides an illuminating interpretation of certain states in the SU_3 coupling scheme. More complicated sets of cluster states will also carry representations of SU_3, but these will in general be reducible. The simple characterization of the irreducible representations of SU_3 mentioned above is restricted to nuclei which can be divided into two clusters, both of which are invariant under SU_3, e.g., closed shells or clusters composed entirely of particles in the 1s shell.

We are indebted to Drs. Wildermuth and Kanellopoulos for sending us a preprint of their forthcoming paper.

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TOTAL CROSS SECTIONS FOR 910 MeV PROTONS

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Abstract: The total cross sections for the interactions of 910 MeV protons with carbon, oxygen, deuterium and hydrogen have been measured. The values obtained are σ_T(carbon) = 362.0±2.4 mb; σ_T(oxygen) = 469±10 mb; σ_T(deuterium) = 79.1±1.0 mb; σ_T(proton) = 46.1±0.5 mb and σ_T("neutron") = 33.0±0.9 mb, where the apostrophes indicate that the neutron was bound in the deuteron. When allowance is made for this, a corrected value of σ_T(neutron) = 39.2±3.1 mb is obtained.

1. Introduction

The experiment reported here was designed to measure the elementary particle total cross sections, σ_T(p—p) and σ_T(p—n). In the first case a carbon-polythene difference measurement was used, except for the smallest angle measurement where the absorber was liquid hydrogen. In the second case a light water-heavy water difference method was employed. From a combination of the two sets of measurements information on the deuterium, carbon and oxygen cross sections was also obtained.

It is not clear to what extent the "p—n" cross section deduced from the D_2O—H_2O measurements will be influenced by the binding of the neutron in the deuteron. Attempts have been made to correct for the "shadow effect" in the deuteron both theoretically and empirically. These will be discussed later.

Total cross section measurements with protons normally encounter difficulties due to multiple and single Coulomb scattering at small angles. This is particularly so for complex nuclei. However, with the focussed proton beam from the Birmingham synchrotron, it was possible to overcome this difficulty partially by using a narrow beam. In addition, knowledge of the differential cross sections for carbon [1]) enabled a more accurate extrapolation to zero angle to be made in the case of carbon and oxygen.

2. Experimental Procedure

2.1. EXPERIMENTAL METHOD

The measurements were made using scintillation counters in the normal transmission arrangement, which is shown in fig. 1. In order to reduce the

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effect of multiple Coulomb scattering, the ratio of the diameter of the back counter to the diameter of the beam at the position of that counter, must be as large as possible [2]). The diameters of the defining counters 1 and 2 were made $\frac{1}{2}''$, while the back counter 3 was 4'' in diameter. The last collimator

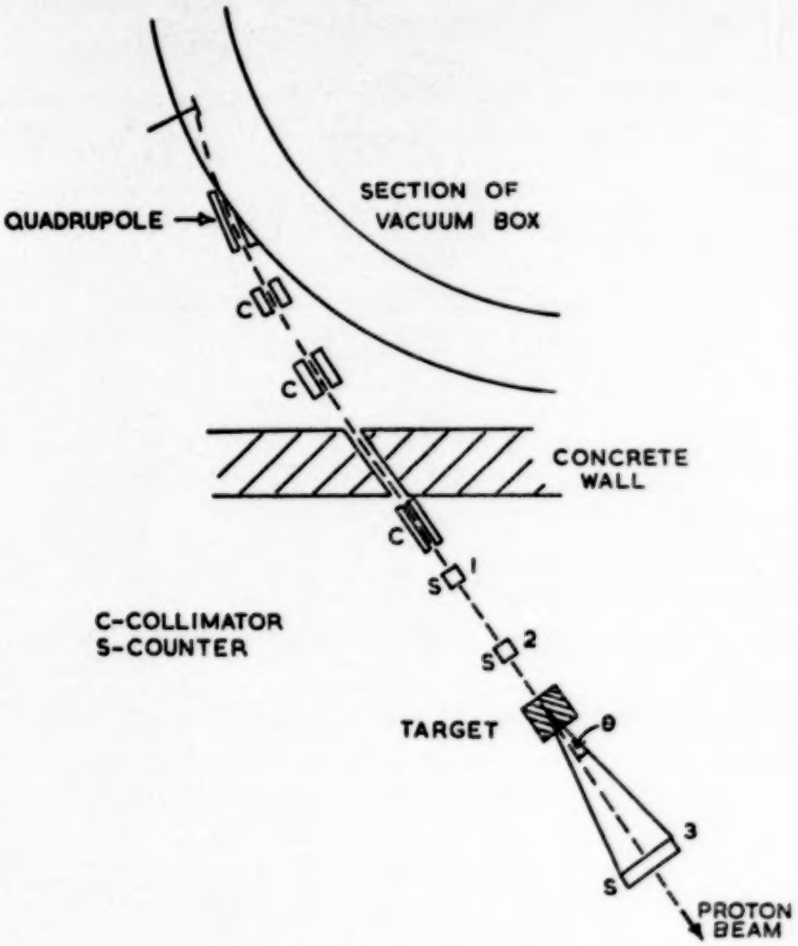


Fig. 1. Experimental arrangement.

was $\frac{1}{2}''$ square in aperture. Conventional fast electronic equipment was used to count double coincidences between counters 1 and 2, and triple coincidences between all three counters. The efficiency of the counter telescope is then defined as the ratio of triple to double coincidence counts during a given time.

The integral cross section at a given half angle θ subtended by the back counter, is given by

$$\sigma(\theta) = \frac{1}{Nx} \log_e \frac{I}{I_0}$$

where I and I_0 are the telescope efficiencies with and without the absorber, N is the number of nuclei per g of absorber, and x is the thickness of absorber in $g \cdot cm^{-2}$.

For $C-CH_2$ and D_2O-H_2O this becomes

$$\sigma(\theta)_{p-p} = \frac{1}{Nx}\left[\log_e I_{CH_2} - a\log_e I_c - (1-a)\log_e I_0\right]$$

$$\sigma(\theta)_{"p-n"} = \frac{1}{N'x'}\left[\log_e I_{D_2O} - a'\log_e I_{H_2O} - (1-a')\log_e I_0\right]$$

where a is the ratio of the number of carbon atoms in the CH_2 absorber to the number in the C absorber, and a' is the equivalent ratio for the D_2O and H_2O absorbers.

Values of $\sigma(\theta)$ were determined at several half angles θ, extrapolation to zero angle then giving the total cross sections.

From considerations of the proton trajectories and energy losses in the defining scintillators etc., the beam energy was estimated to be $907 \pm \frac{23}{30}$ MeV.

2.2. CORRECTIONS

The values of $\sigma(\theta)_{carbon}$ and $\sigma(\theta)_{p-p}$ were liable to be too high at small angles due to

(i) single Coulomb scattering,

(ii) multiple Coulomb scattering.

For the D_2O-H_2O measurements these effects were equal and no correction was necessary.

(i) *Single Coulomb scattering*

For carbon the differential Coulomb cross section was calculated from the expression for the scattering from a charged sphere of radius R, i.e.

$$\frac{d\sigma_c}{d\Omega} = \left(\frac{Ze^3}{2p\beta c}\right)^2 \times \frac{1}{\sin^4 \tfrac{1}{2}\theta} \times \frac{1}{[1 + kR\sin^2 \tfrac{1}{2}\theta]^2}$$

where Z is the atomic number, e is the electronic charge, p, βc and k are the momentum, velocity and wave number respectively of the incident proton, and θ is the angle of scatter. This differential cross section was then integrated to give the Coulomb cross section σ_C at the relevant angles. The corrections were negligible for all but the four smallest solid angles, and then did not amount to more than 2 % of the total.

For hydrogen, because of the identity of the two particles, the above expression must be modified [3]). The correction was negligible for the $C-CH_2$ measurements but was of the order of 1 % for the liquid hydrogen determination.

No account was taken of any possible interference between nuclear and Coulomb scattering at small angles. It would appear that this is small at this energy in the case of carbon [4]), and would have a negligible effect on the integrated cross sections. There are no data available for hydrogen at this

energy, but Mescheryakov, Nurushev and Stoletov [5]) find that at 635 MeV the interference term is extremely small if not zero.

(ii) *Multiple Coulomb scattering*

Multiple Coulomb scattering gives rise to an apparent cross section, for a particular absorber thickness,

$$\sigma_m = \frac{1}{Nx} \log_e F$$

where $(1-F)$ is the fraction of the beam multiply scattered out of counter 3.

Because of the different number of carbon atoms in the CH_2 and C absorbers, the expression becomes in this case

$$\sigma_m = \frac{1}{Nx} [\log_e F_{CH_2} - a \log_e F_C].$$

With a knowledge of the counter geometry and the beam diameter at the position of counter 3, the values of F could be found from the curves of Sternheimer [2]). The corrections were only appreciable for the two smallest angle measurements in both the carbon and (p—p) determinations. The errors on these measurements have been increased to take into account the uncertainty in the corrections.

2.3. ERRORS

The statistical standard deviations on the measured values of $\sigma(\theta)$ were calculated from the formulae of Chen, Leavitt and Shapiro [6, 6a]). Additional errors could arise from accidental counts and 'pile-up' in the electronics, but these were estimated to be negligible. Errors due to long term drifts in counting efficiency were eliminated by alternating short 'absorber out' and 'absorber in' runs.

3. Results for Carbon

The corrected values of $\sigma(\theta)$ are plotted against half angle θ and solid angle Ω in figs. 2a and 2b. In previous experiments, (e.g. Booth, Ledley, Walker and White [7])) the total cross section has been determined by assuming a linear relationship between $\sigma(\theta)$ and θ, and extrapolating to zero angle. Such an assumption becomes increasingly dubious at high energies. A linear relationship with Ω only exists if the angular distribution is constant over the range of angles covered. For carbon this is certainly not so, for the diffraction scattering is strongly peaked forward at small angles. It is obvious from figs. 2a and 2b that a linear extrapolation will give a high value of σ_T in the case of θ, and a low value in the case of Ω.

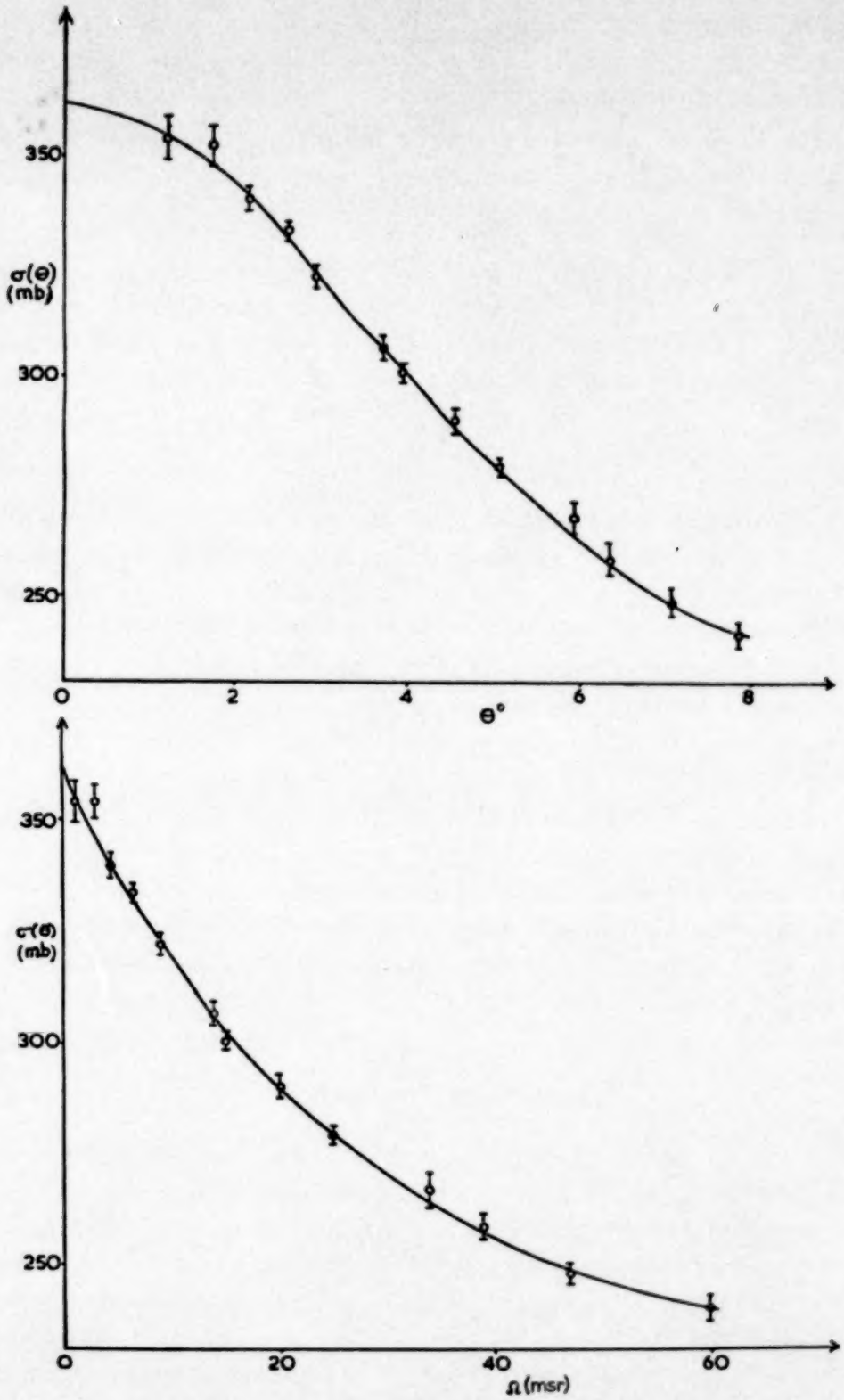


Fig. 2a. Integral cross sections for carbon, 910 MeV, (against θ).
Fig. 2b. Integral cross sections for carbon, 910 MeV, (against Ω).

The angular distribution has been determined for carbon at this energy, during an experiment to measure the polarisation produced in the second scatter of protons from carbon [8]). This angular distribution is given in fig. 3, the ordinate being arbitrary as there was no absolute monitor during the

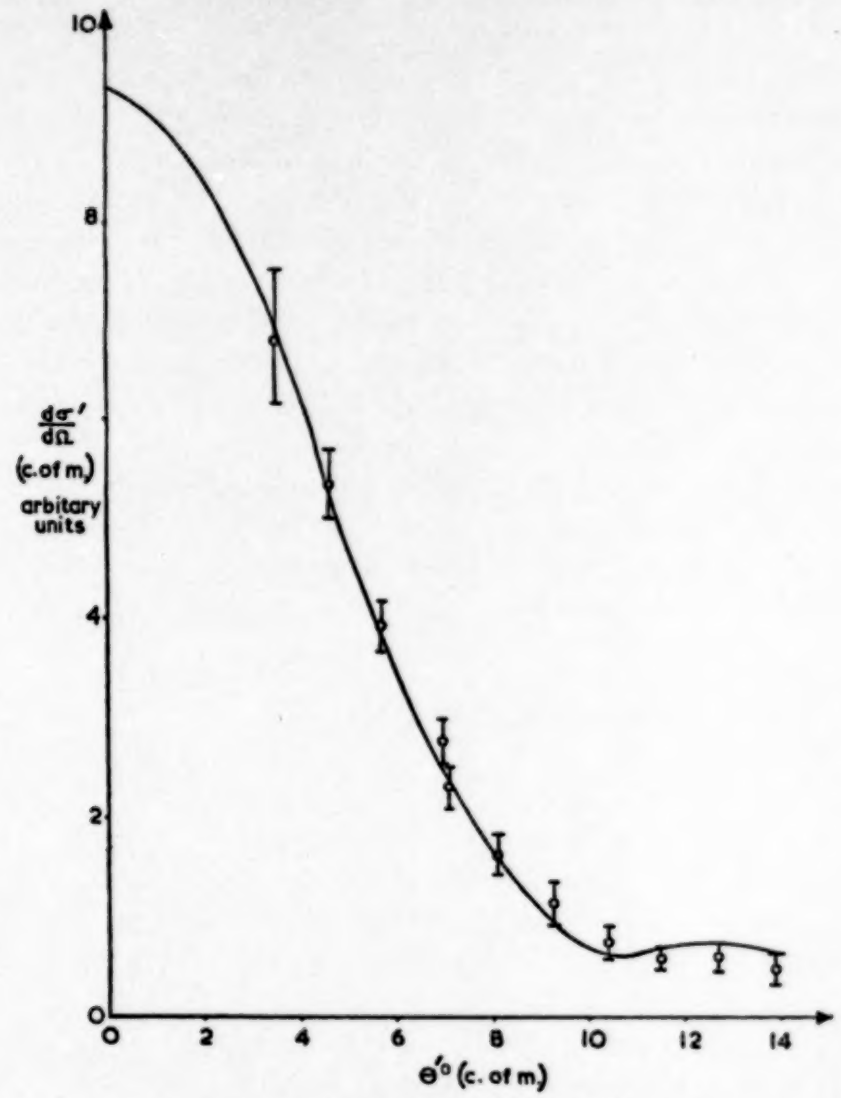


Fig. 3. Angular distribution for carbon, 950 MeV, (from polarization data).

experiment. The curve drawn in fig. 3 was calculated by assuming that the diffraction scattering is given by the distribution for a 'black disc', and that the inelastic scattering is isotropic, i.e.

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{total}} = N\left[A\left|\frac{J_1(kR\sin\theta)}{\sin\theta}\right|^2 + B\right]$$

where $A\,(=\sigma_d/\pi)$ and B are constants, N is a normalising constant and J_1 is a first order Bessel function. Several curves with different values of kR, A and B were fitted to the experimental points, and the best curve deter-

mined by the method of least squares. Despite the simplified assumption for $d\sigma/d\Omega$, the fit to the experimental points is good. However, Brown, Ashmore and Nordhagen [9]) have shown that the 'black disc' distribution, with suitable variation of the radius parameter, can be made to agree with several different optical model potentials; at least at angles smaller than the first diffraction minimum.

The calculated distribution of fig. 3 was then integrated numerically from 60 msr to 0 msr, and normalised to fit the values of $\sigma(\theta)$ in figs. 2a and 2b, assuming a value of $\sigma(\theta) = 240$ mb at 60 msr. The curves drawn are the best fits obtained by the method of least squares, and give a value of $\sigma_T = 362.0 \pm 2.4$ mb. The quoted error includes the standard deviation from the least squares fit, and an allowance for any error in the choice of kR in the equation for $d\sigma/d\Omega$. The best value of kR was found to be 21.5 (in laboratory system), but fitting the distribution with a value of 20.5 only led to a change of 1 % in σ_T. It was unlikely that the value of kR was in error by this amount.

Once the integrated angular distribution was normalised to the absolute values of $\sigma(\theta)$, it was possible to make the original angular distribution absolute, and to determine the values of the forward scattering amplitude $f(0)$, and the constant $A\pi$ in the 'black disc' expression. The values obtained were

$$|f(0)|^2 = \frac{d\sigma}{d\Omega}(0) = 5.09 \pm 0.32 \text{ b/sr},$$

$$A\pi = 130.6 \pm 3.0 \text{ mb}.$$

These values are in good agreement with the data of Huq *et al.*[1]), obtained from an absolute determination of the angular distribution. The value of $|f(0)|^2$ is expected to contain a small contamination due to inelastic scattering in the forward direction. However, Huq *et al.* found a negligible contribution at angles below 4°. The contribution from the constant B in the expression for $d\sigma/d\Omega$ amounts to 0.20 b/sr at 0°.

The constant $A\pi$ is nominally equal to the diffraction cross section σ_d. However, the 'black disc' model gives of the order of 15 % of the diffraction scattering beyond the first diffraction minimum, whereas Musgrave [10]), from bubble chamber exposures, finds approximately 7 % in this region. The value of σ_d obtained by correcting $A\pi$ to take into account this discrepancy was

$$\sigma_d = 120.1 \pm 8.0 \text{ mb}.$$

The error has been increased to allow for the uncertainty in this correction. Subtraction of σ_d from σ_T leads to a value of 241.9 ± 8.4 mb for the absorption cross section σ_a. The errors include the effect of a possible wrong choice of kR, as for σ_T.

Fig. 4 shows the measured values of σ_T, σ_d and σ_a compared with those of other experiments at energies from 200 MeV upwards.

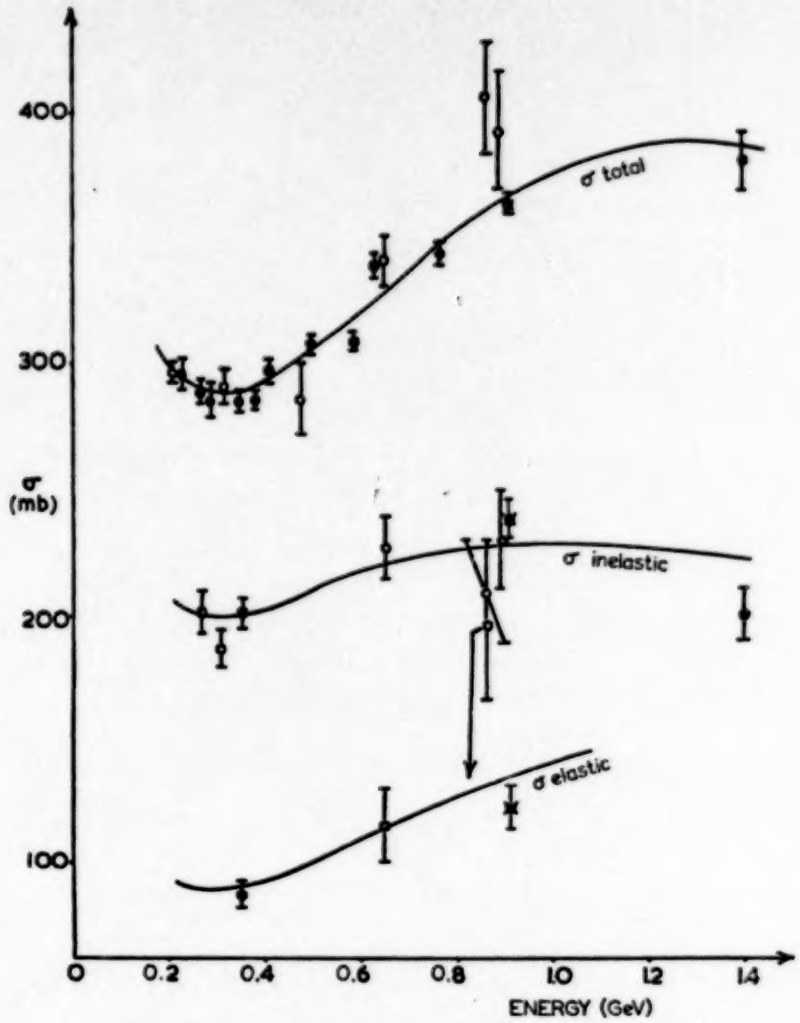


Fig. 4. Energy variation of cross sections, carbon.
○ protons,
● neutrons,
✱ this experiment.

It is possible to calculate the imaginary part of the forward scattering amplitude from the value of σ_T using the optical theorem

$$[\mathrm{Im}\, f(0)]^2 = \left(\frac{k\sigma_T}{4\pi}\right)^2 = 5.41 \pm 0.13 \text{ b/sr}.$$

We obtained a value of 5.09 ± 0.32 b/sr for $|f(0)|^2$ therefore the ratio

$$\frac{[\mathrm{Im}\, f(0)]^2}{|f(0)|^2} = 1.06 \pm 0.07.$$

This indicates that at this energy the real part of the scattering amplitude is extremely small.

4. Results for p—p and "p—n"

The values of $\sigma(\theta)_{p-p}$ obtained from C—CH_2 and liquid hydrogen, and $\sigma(\theta)_{"p-n"}$ obtained from D_2O—H_2O are plotted against Ω in figs. 5a and 5b.

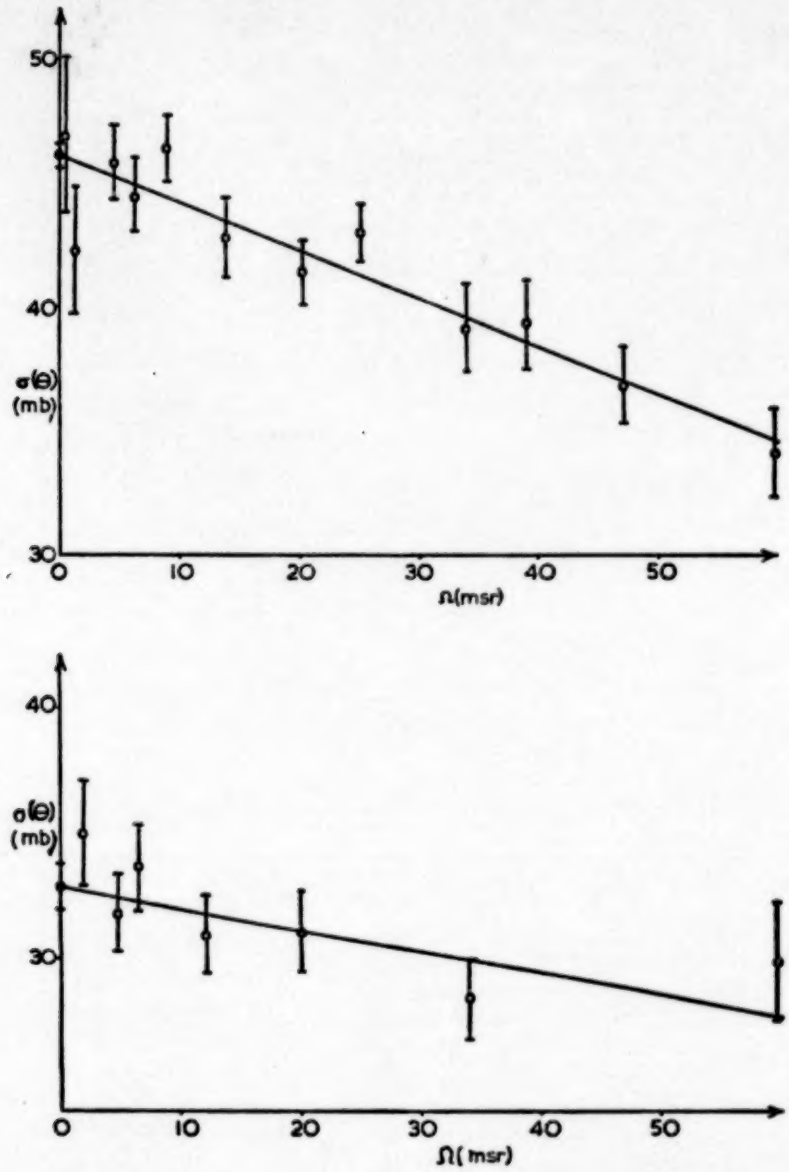


Fig. 5a. σ_θ(p—p) 910 MeV.
Fig. 5b.σ_θ("p—n") 910 MeV.

In these cases a linear extrapolation is a good approximation, since although both the elastic and inelastic differential cross sections are peaked forward, they are not changing rapidly in this angular range. It was estimated that with the particular geometry used the error in the extrapolation was not

likely to be greater than 0.25 % for σ_T(p−p) and 0.5 % for σ_T("p−n"). The linear extrapolation gives at 910 MeV

$$\sigma_T(\text{p}-\text{p}) = 46.1 \pm 0.5 \text{ mb},$$
$$\sigma_T(\text{"p}-\text{n"}) = 33.0 \pm 0.9 \text{ mb}.$$

The value of σ_T("p−n") cannot be compared directly with σ_T(n−p) because a correction must be made at this energy for the shielding of one nucleon in the deuteron by the other. This shadowing effect appears to be zero at $\approx$ 400 MeV, but rises to the order of 9 mb at 1.4 GeV [11, 6a]). Glauber [12]) and Blair [13]) have calculated the expected magnitude of the effect assuming that the nucleons may be represented as purely absorbing spheres. However, as they have pointed out, there are uncertainties in these calculations due to

(i) Lack of knowledge of the deuteron wave function especially at small distances,

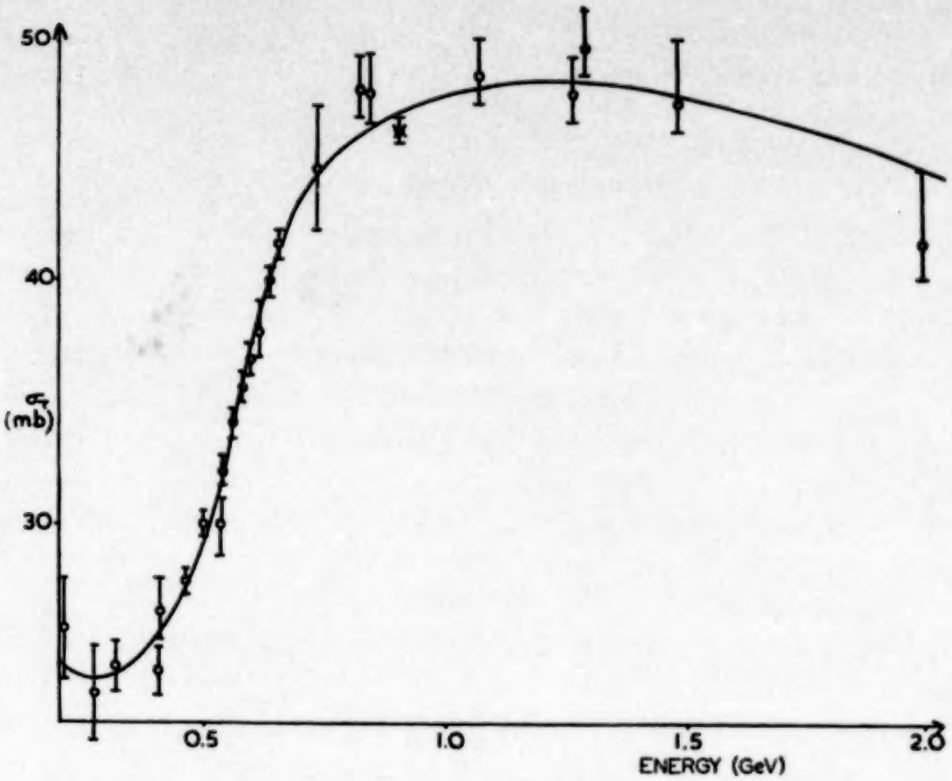
(ii) the assumption of 'black' nucleons.

At 910 MeV a direct estimate of the correction could be obtained from the measurements of Batson, Calwick, Klepp and Riddiford [14]) who used a deuterium filled cloud chamber in a 970 MeV proton beam. Comparison with the value of Batson *et al.* for σ_T("p−p") with the measured σ_T(p−p) at 910 MeV gave a proton-proton defect of 3.1 ± 2.1 mb. Assuming that the neutron-proton defect is equal to this, the total correction is 6.2 ± 2.8 mb. The corrected value of σ_T("p−n") is then equal to 39.2 ± 3.1 mb.

The values of σ_T(p−p) and σ_T(p−n) at 910 MeV are compared with measurements at other energies in figs. 6a and 6b. The value of σ_T(p−n) is in agreement with the corresponding value of Batson *et al.* at 970 MeV, and with the available measurements of σ_T(n−p). It is higher than the values of σ_T("p−n")$_{\text{corr.}}$ of Chen *et al.*, but this may be partially due to the different methods of correcting σ_T("p−n").

The total n−p cross section is varying somewhat differently from the p−p cross section. Although there is a corresponding dip at $\approx$ 400 MeV, the rise up to 1 GeV, resulting from meson production, is less steep in the case of n−p. These differences are not unexpected if one considers the available isobaric spin states for the respective interactions. While the p−p interaction can only occur in the $T = 1$ isobaric spin state, the n−p interaction occurs both in $T = 0$ and $T = 1$ states. Fig. 7 shows the variation of $\sigma(T = 1)$ (i.e. σ_T(p−p)) with energy, and the values of $\sigma(T = 0)$ calculated from the available measurements of σ_T(n−p). $\sigma(T = 0)$ has also been calculated from the values of σ_T(p−p) and σ_T(p−n) at 910 MeV, and from the results of Batson *et al.* at 970 MeV. The curve for $\sigma(T = 0)$ was given by Chen *et al.* from their values of σ_T(p−p) and σ_T("p−n")$_{\text{corr.}}$.

As a direct consequence of the smaller value of σ_T(p−p) and the larger value of σ_T(p−n), the value of $\sigma(T = 0)$ at 910 MeV is much larger than the



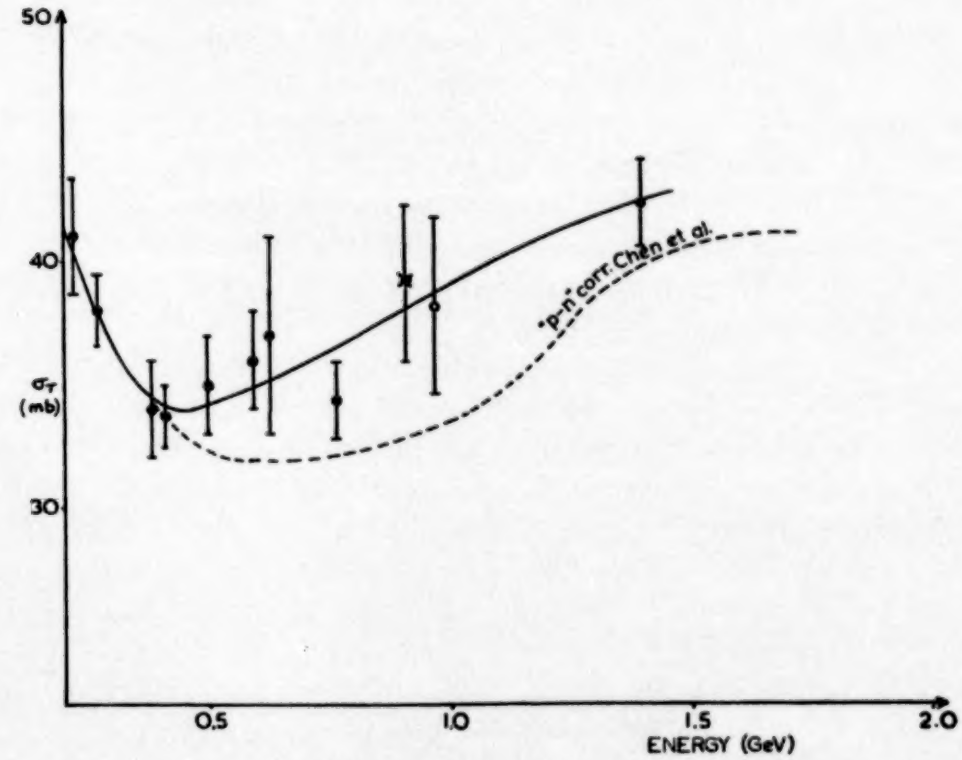


Fig. 6a. Variation of σ_T (p−p) with energy.
* this experiment.

Fig. 6b. Variation of σ_T (n−p) with energy.
● n−p,
○ p−n,
* this experiment.

corresponding value of Chen *et al.* However, within errors, it is in agreement with the variation suggested by the other values of $\sigma(T = 0)$, with the possible exception of the 765 MeV point obtained from the results of Booth, Hutchinson and Ledley [15]). It is possible that the dip in the $\sigma(T = 0)$ curve

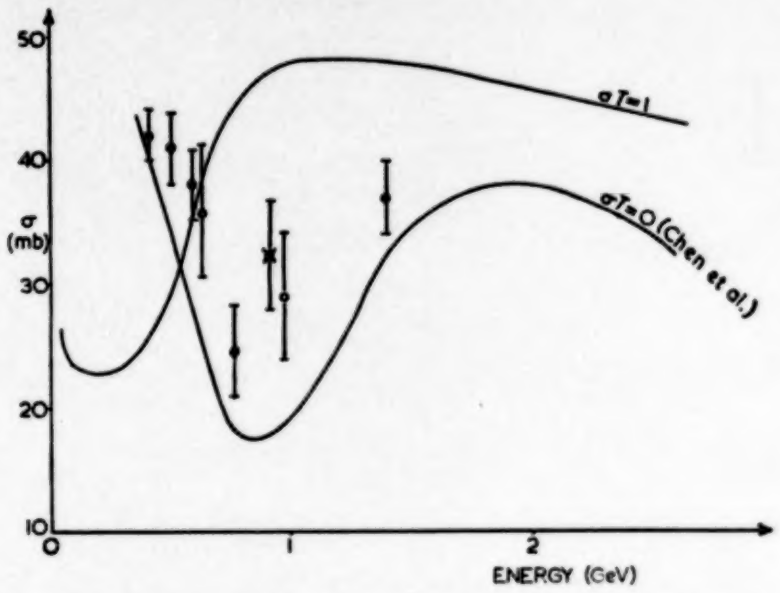


Fig. 7. Curves representing the variation of $\sigma(T=0)$ and $\sigma(T=1)$ with energy.

$\sigma(T = 0)$ { ● from n−p, ○ from "p−n" corr., ✱ this experiment.

at ≈ 1 GeV is not so pronounced as that suggested by Chen *et al.*, but nevertheless the curve follows the same general shape as the $\sigma(T = 1)$ curve, but displaced to higher energies by approximately 700 MeV.

It is tempting to describe the variation in $\sigma(T = 0)$ and $\sigma(T = 1)$ in terms of the $T = \frac{3}{2}$ isobar theory of pion production, as was suggested by Chen *et al.* However, it is the variations of the inelastic contributions to $\sigma(T = 0)$ and $\sigma(T = 1)$ which are of importance, and the description of Chen *et al.* is only valid if the elastic contributions are varying in a similar manner. It would appear [14]) that this is not so, and in fact the inelastic $\sigma(T = 0)$ is of the same order of magnitude as the inelastic $\sigma(T = 1)$ below 1 GeV.

5. Results for Deuterium and Oxygen

Since both σ_θ(p−p) and σ_θ("p−n") against Ω are good approximations to straight lines, the total deuterium cross section may be obtained from the addition of σ_T(p−p) and σ_T("p−n"). The value obtained was 79.1 ± 1.0 mb.

The corrected values of σ_θ(oxygen) obtained from the difference in $\sigma_\theta(H_2O)$ and $2\,\sigma_\theta$(p−p), are plotted against θ and Ω in figs. 8a and 8b.

It is again obvious that neither case is a good approximation to a straight line. Since there is no direct information available on the angular distribution for oxygen at this energy, it was assumed that it could be fitted by a

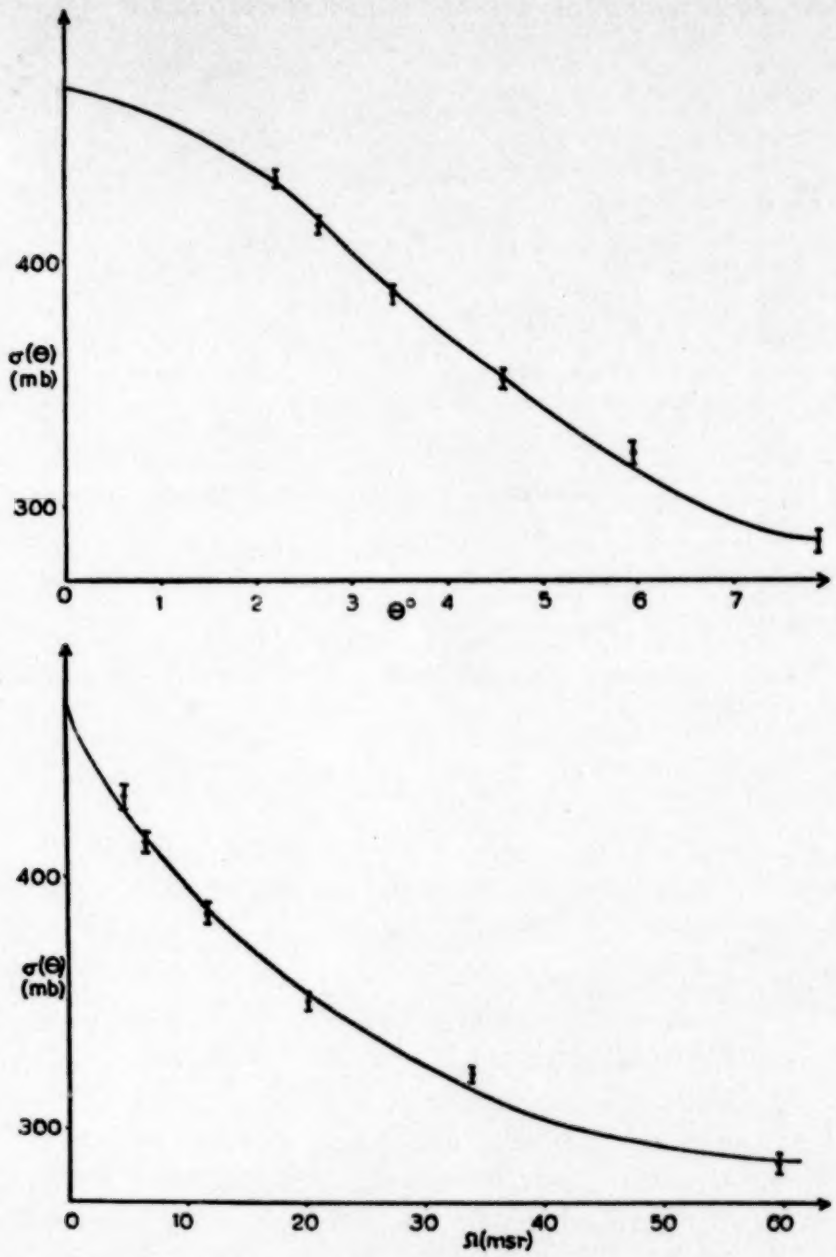


Fig. 8a. Integral cross sections for oxygen, 910 MeV, (against θ).
Fig. 8b. Integral cross sections for oxygen, 910 MeV, (against Ω).

'black disc' expression similar to that used for carbon. The relevant value of kR for oxygen (23.5 in laboratory system), was deduced from the value used for carbon and the expression $R = r_0 A^{\frac{1}{3}}$. As it is unlikely that there will be any radical change in the constants of $d\sigma/d\Omega$ for two nuclei of similar atomic number, the values of the constants A and B were taken to be the same as

for carbon. The values of $\sigma(\theta)$ were then fitted with a normalised integrated angular distribution to give a value of $\sigma_T = 469 \pm 10$ mb. The error given by the least squares fit was increased to take into account the uncertainty in the choice of the shape of $d\sigma/d\Omega$. It was also possible to calculate a value of σ_d and hence σ_a as for carbon. The values obtained were

$$\sigma_d = 174 \pm 16 \text{ mb}, \qquad \sigma_a = 295 \pm 19 \text{ mb}.$$

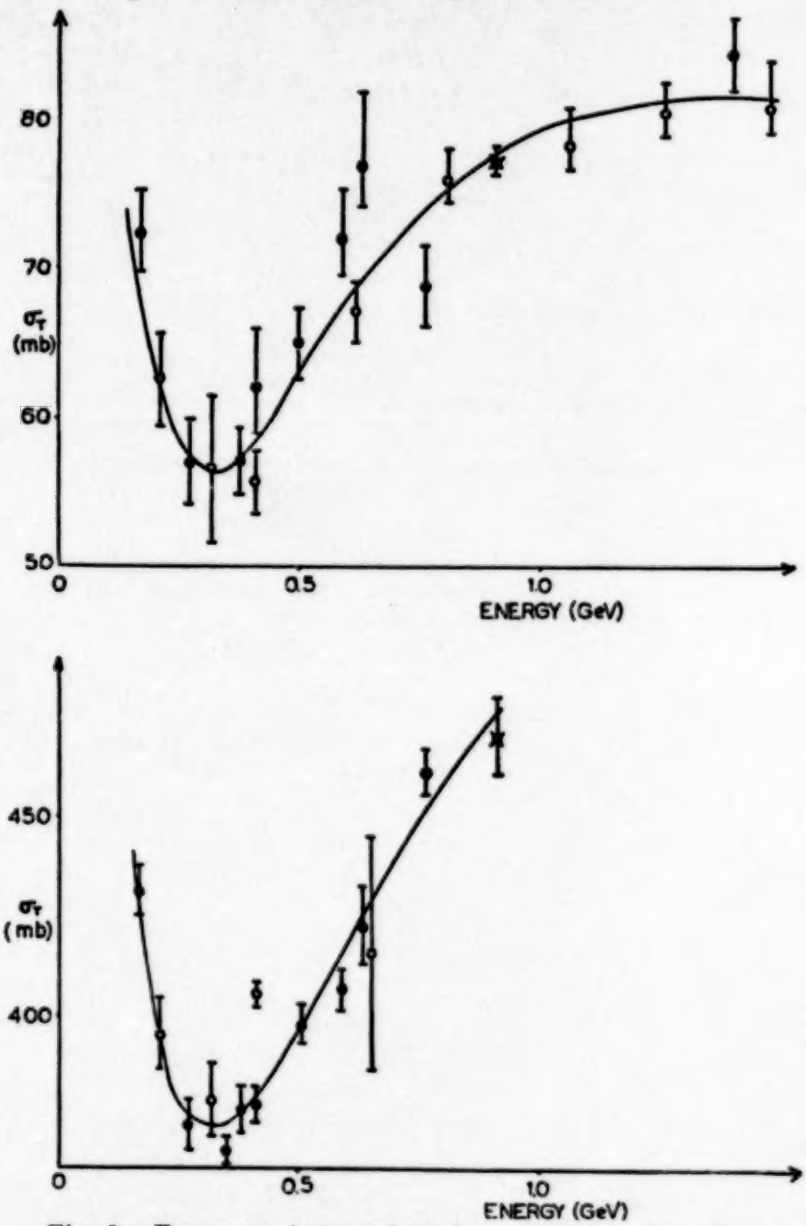


Fig. 9a. Energy variation of total cross section, deuterium.
● neutrons,
○ protons,
✱ this experiment.

Fig. 9b. Energy variation of total cross section, oxygen.
● neutrons,
○ protons,
✱ this experiment.

Figs. 9a and 9b show the values of σ_T(deuterium) and σ_T(oxygen) at 910 MeV compared with other measurements at energies in this range.

We wish to express our thanks to the other members of the synchrotron counter group, and in particular to Mr. A. M. Segar, for their assistance during synchrotron runs. We are also grateful to Professor P. B. Moon for his encouragement throughout this work.

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THE EFFECTIVE NUCLEAR POTENTIAL

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Abstract: An empirical analysis is made of the mean effective internucleon potential required in the shell-model description of nuclei, allowing for the presence of many-body effects as suggested by current theory. A consistent description is found in which the effective two-body interaction acts almost entirely in even states, and the many-body effects are simulated by a repulsive three-body contact interaction. The strength of the two-body interaction is consistent with that expressed by the free scattering matrix of the two-nucleon system, and that of the three-body interaction with the 'rearrangement energy' calculated in the many-body theory.

1. Introduction

In the decade that has elapsed since the publication of Rosenfeld's study of nuclear forces [1]) there have been several developments of great significance for the problem of nuclear forces in relation to the structure of nuclei. The shell model [2]) of nuclei has acquired a firm empirical standing, based on the multiplicity of facts that can be explained with the assumption that nucleons move in independent orbits inside a nucleus, generalised, in the more complex cases, to transmigration between a relatively small number of orbits. The character of the single-particle potential needed to define these orbits is known to be that of a rounded square well with a strong attractive spin-orbit coupling; the occupancy of nearly degenerate states is determined by residual interactions, whose form and strength are similar to those effective in the deuteron.

The detailed nature of this interaction is unimportant for many results; usually a simple radial shape has been chosen for the potential, gaussian or Yukawa, and an exchange character has been taken either to fit the facts as well as possible in the region considered or such as to describe the low-energy properties of the two-nucleon system; it has often been assumed that the interaction must have the 'symmetrical' character, suggested by second-order meson theory, to ensure nuclear saturation. It is now clear that such an interaction is very different from the real potential acting between two (free) nucleons. The form of this real potential is still uncertain, but recent analyses [3,4]) of nucleon-nucleon scattering in terms of a potential indicate the presence of strong non-central forces and of a 'repulsive core' at small distances.

This last feature especially has stimulated a critical examination of the theory of many-body systems and of the foundations of the empirical shell-model structure; a review has been made by Eden [5]). At an early stage in this investigation it was suggested by Brueckner [6]) that the effective potential used in shell-model calculations could be approximated by the scattering matrix of the two-nucleon system, generalised off the energy shell when necessary. This simple identification is not now believed to give a reasonable first approximation; it is necessary to allow for the average effects of the presence of other nucleons on the scattering of any one pair, through the mean potential they provide and through the effects of the exclusion principle. If it is possible then to construct an effective interaction that can be handled in the simple manner of shell-model calculations it will not necessarily possess also the attributes of a 'realistic' two-body interaction; it will not necessarily describe the deuteron system adequately, nor will nuclear saturation be related only to the exchange character.

The purpose of this paper † is to re-examine the empirical evidence about the potential effective in complex nuclei in the light of the current discussions of the many-body system; a preliminary account of this work has been given elsewhere [7]). It is now customary to speak of the shell-model system, described by a wave-function Φ, as providing a *model* of the real system described by the wave-function Ψ; their relation is described symbolically by

$$\Psi = F\Phi \tag{1}$$

where F introduces all the local correlations characteristic of the actual system. If a correspondence is established between the real states Ψ_n and an orthonormal set Φ_n, the effective Hamiltonian may be defined as that which has the eigenfunctions Φ_n with the eigenvalues E_n equal to those of the real system.

If the effective Hamiltonian is defined in this formal way it is evident that the usual variational principle is applicable, and may be used to determine the ground-state model wave-function in the Hartree-Fock manner. For practical use, however, we must make the additional assumption that the effective Hamiltonian can be expressed in a comparatively simple way in terms of internucleon potentials; the empirical test of this hypothesis is the main purpose of the present paper.

Two kinds of assumption have to be made in this analysis, the form of the Φ_n and the form of the effective interaction. The 'model' for infinite nuclear matter is taken, as usual, to be a Fermi gas; for closed shell nuclei, and those adjacent, nuclear theory suggests a wave-function describable by a single Slater determinant, that is unique apart from the unknown shape of the

† The present paper, planned as the first of a series, is confined to discussion of the *mean* potential.

single-particle orbital functions. But for nuclei further removed from closed shells the model should probably include long-range correlation effects, such as intrinsic distortion or surface vibrations.

The effective interaction potential will *not* be the same as that defined in the self-consistent many-body theory (to which the variational principle is not applicable). The potential used in our analysis must contain three-body, and generally many-body, terms which describe the way in which interaction between two particles is influenced by the presence of others; the two-body terms alone should be related closely to the scattering between free nucleons.

These considerations have led to the following ansatz for the form of the effective potential:

$$T = \sum_{i<j}\sum t_{ij} + \sum_{i<j<k}\sum\sum t_{ijk} \tag{2}$$

in which the many-body effects have been simulated by three-body terms alone, for the sake of simplicity of calculation. From the view-point of many-body theory there is no reason why the two-body terms should have the form of a static potential though with our philosophy they should be functions of the *relative* coordinates of the nucleons; for the sake of a simple comparison with the scattering matrix we have expressed them as

$$t_{12} = \delta(\mathbf{r}_1 - \mathbf{r}_2)t(\mathbf{k}', \mathbf{k}) \tag{3}$$

where $\mathbf{k}$ is the operator corresponding to the relative wave-number,

$$\mathbf{k} = \tfrac{1}{2}i(\nabla_1 - \nabla_2); \tag{4}$$

placed on the *right* of the delta-function $\mathbf{k}'$ denotes the same operator placed on the *left*; this form was used in an earlier discussion [8]) of the spin-orbit potential. An ordinary static (Wigner) interaction would then be described by a t dependent only on $\mathbf{k}' - \mathbf{k}$.

It is generally believed that the most important part of the two-body interaction can be represented by a contact potential, i.e. by constant $t(\mathbf{k}', \mathbf{k})$; this suggests an expansion in powers of $\mathbf{k}'$ and $\mathbf{k}$. If this expansion is stopped at the quadratic terms only a small number of undetermined coefficients occur, and an attempt can be made to determine these by comparison with experimental energies. However, this can only be done in conjunction with a similar determination of the parameters of the three-body interaction, and to reduce the problem to manageable proportions only the simplest possible form for the latter has been considered, the contact interaction

$$t_{123} = \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_3 - \mathbf{r}_1)t_3 \tag{5}$$

with one undetermined constant t_3. Averaged over one of the particles this gives a contribution to the two-body contact interaction proportional to the local density. Some discussion of this assumption has been given by Squires [9])

This form of potential is given in more detail in the following section. In the approximation which has been most extensively investigated there are four undetermined constants involved in the 'mean' potential (i.e. the average involved in nuclei with equal numbers of protons and neutrons, each with their spins equally in both directions). These have been fixed, approximately, by considering the properties of infinite nuclear matter (section 3), including surface effects, and of light nuclei near closed shells using oscillator wave-functions for those of the single-particles (section 4), with the results given in section 5, which are the same as those previously reported [7]). The spin-orbit part of the interaction cannot be adequately described by the extreme short-range approximation and will be discussed separately in a following paper [10]). The remaining parts of the interaction, i.e. the tensor components and spin-dependent parts of the central potentials, are much more difficult to determine unambiguously. In principle they may be inferred from the level structure of nuclei more distant from closed shells and earlier estimates were based upon a study of the configuration $(s^4 p^n)$ for nuclei between He and O; we now believe however, that even in this case, in the p-shell, distortion may introduce configuration interaction effects that should more naturally be included in the model wave-function than in the effective potential [11]). For some estimates made in this paper the spin-dependence of the central interaction has been represented by a single term.

We have also considered briefly some simple generalisations of the form assumed for the mean potential (sections 6 and 7); they do not appear to alter the qualitative features previously found. Finally, in sections 8 and 9, we compare the strength-constants of our potential with other available evidence.

2. Assumed Potential

To complete the specification of the potential given by equations (2) to (5) above we must introduce an explicit form for $t(\mathbf{k}', \mathbf{k})$. A polynomial expansion in powers of $\mathbf{k}$ and $\mathbf{k}'$ has been chosen on account of the comparatively simple algebraic expressions for the matrix elements; on the other hand this form is unrealistic for large momentum transfers, so that it is not suitable for the discussion of second-order effects, unless some momentum cut-off is introduced. If the polynomial is limited to terms quadratic in the momenta (analogous to the effective mass approximation), the most general form is

$$\begin{aligned} t(\mathbf{k}', \mathbf{k}) = {} & t_0(1+x_0 P^\sigma)+\tfrac{1}{2}t_1(1+x_1 P^\sigma)(\mathbf{k}'^2+\mathbf{k}^2) \\ & + t_2[1+x_2(P^\sigma-\tfrac{4}{5})]\mathbf{k}'\cdot\mathbf{k} \\ & + \tfrac{1}{2}T[\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\mathbf{k}^2+\text{conj.}] \\ & + \tfrac{1}{2}U[\boldsymbol{\sigma}_1\cdot\mathbf{k}'\boldsymbol{\sigma}_2\cdot\mathbf{k}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\mathbf{k}'\cdot\mathbf{k}+\text{conj.}] \\ & + V[i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}'\times\mathbf{k}], \end{aligned} \tag{6}$$

where P^σ is the spin-exchange operator (in this form it is unnecessary also to introduce isobaric spin-dependence). The constants t_0, t_1, t_2 are a measure of the *mean* central potential, with an exchange character specified by x_0, x_1, x_2; T and U characterise a tensor potential in even and odd states respectively, and V describes a short range two-body spin-orbit potential.

The validity of this approximation to $t(\mathbf{k}', \mathbf{k})$ is considered only from the empirical point of view in our analysis. It has appeared from our study [10]) of the spin-orbit coupling that the extreme short-range V-term is quite inadequate to explain the observed variations of spin-orbit coupling among nuclei, and this may also be true of the tensor interactions; however, this does not affect the conclusions of the present paper, which is concerned primarily with the mean potential.

It is convenient to describe here some additional terms that are considered briefly in sections 6 and 7. The first is suggested by the presence of considerable D-waves in the nucleon-nucleon interaction at energies around 100 MeV, and is taken in the form

$$t_D[\mathbf{k}^2\mathbf{k}'^2-(\mathbf{k}\cdot\mathbf{k}')^2]. \tag{7}$$

The second is a four-body interaction in T, which is again assumed to have zero range:

$$t_4\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}_4). \tag{8}$$

The expectation value of the two-body interaction (6) has to be calculated for an antisymmetrised wave-function; when a *product* wave-function is used t must be multiplied by the antisymmetrising factor $(1-P^xP^\sigma P^\tau)$, so that the effective interaction becomes

$$t(\mathbf{k}', \mathbf{k})-t(\mathbf{k}'-\mathbf{k})P^\sigma P^\tau. \tag{9}$$

The *mean* potential considered in this paper is the average of (9) over spins and isobaric spins, which reduces to

$$\tfrac{3}{4}t_0+\tfrac{3}{8}t_1(\mathbf{k}^2+\mathbf{k}'^2)+\tfrac{5}{4}t_2\mathbf{k}'\cdot\mathbf{k}. \tag{10}$$

Similarly in (7), t is replaced by $\frac{3}{4}t$; the same averaging applied to the three- or four-body interactions (5) and (8), results in the multiplication of t_3 by the factor $\frac{3}{8}$, and t_4 by $\frac{3}{32}$.

This mean potential is also appropriate for systems differing by one particle from symmetrical ones, apart from an additional contribution from the spin-orbit term.

3. Infinite Nuclear Matter

We consider first idealised infinite nuclear matter, described by a Fermi gas with spin and isobaric spin states equally occupied. This is a characterised by a density ρ_0 and a mean binding energy per particle B; their values

have been taken to be those given by Cameron [12]) in an analysis of the semi-empirical mass-formula †. In terms of a radius constant r_0, and a convenient unit of energy $E_0 = 1/(2Mr_0^2)$,

$$r_0 = 1.112\times10^{-13}\,\text{cm}, \qquad E_0 = 16.80\,\text{MeV},$$

one has

$$\rho_0 = (\tfrac{4}{3}\pi r_0^3)^{-1}, \qquad B/E_0 = 1.014. \tag{11}$$

The expectation value of the mean effective Hamiltonian for such a Fermi gas gives

$$-B/E_0 = 1.38(1+\lambda_1+\lambda_2)+\lambda_0+\lambda_3+\lambda_4 \tag{12}$$

where the dimensionless parameters λ_n are defined by

$$\begin{aligned}
\lambda_0 &= \tfrac{1}{2}(\rho_0/E_0)\tfrac{3}{4}t_0\\
\lambda_1 &= \tfrac{1}{4}(\rho_0/E_0r_0^2)\tfrac{3}{4}t_1\\
\lambda_2 &= \tfrac{1}{4}(\rho_0/E_0r_0^2)\tfrac{5}{4}t_2\\
\lambda_3 &= \tfrac{1}{6}(\rho_0^2/E_0)\tfrac{3}{8}t_3\\
\lambda_4 &= \tfrac{1}{24}(\rho_0^3/E_0)\tfrac{3}{32}t_4.
\end{aligned} \tag{13}$$

The condition that B be stationary for variations of ρ_0 gives

$$0 = 1.38\,(\tfrac{2}{3}+\tfrac{5}{3}\,(\lambda_1+\lambda_2))+\lambda_0+2\lambda_3+3\lambda_4. \tag{14}$$

Throughout the following analysis these relations are used to eliminate two of the parameters λ_n; it is convenient to eliminate the most important parts of the potential, which are in fact those specified by λ_0 and λ_1;

$$\begin{aligned}
\lambda_0 &= -4.605+0.5\lambda_3+2.0\lambda_4,\\
\lambda_1 &= 1.60-\lambda_2-1.09(\lambda_3+2\lambda_4).
\end{aligned} \tag{15}$$

The interaction given by (7) does not affect the forward or backward scattering and does not therefore contribute to the 'infinite' case.

The self-consistent single-particle potential, defined by the Hartree-Fock approximation has a quadratic momentum dependence, on account of the assumption (6), and may be described by an effective mass M^* and a well-depth V_0 (for the lowest state), where

$$\begin{aligned}
M/M^* &= 1+\lambda_1+\lambda_2 = 2.60-1.09(\lambda_3+2\lambda_4),\\
V_0/E_0 &= 1.38(\lambda_1+\lambda_2)+2\lambda_0+3\lambda_3+4\lambda_4 = -7.00+2.5(\lambda_3+2\lambda_4).
\end{aligned} \tag{16}$$

The energy of the topmost particle is $V_0+2.30(M/M^*)E_0 = -B$.

The surface effects in a large nucleus may be estimated in the way suggested by us in an earlier analysis [13]). In that investigation we used a simple analytic approximation to the form of the energy as a function of density; we have verified (for the values of the parameters finally selected) that this is indeed a close approximation. The density distribution, according to the

† Different values of these parameters were also considered, but consistency with light nuclei could only be obtained for values within 3 % of those quoted.

Thomas-Fermi approximation has the form †

$$\rho/\rho_0 = [\tanh(R-r)/2b]^2 \tag{17}$$

where, in the present notation,

$$b^2/r_0^2 = \tfrac{3}{4}\lambda_1 - \tfrac{1}{4}\lambda_2 = 1.20 - \lambda_2 - 0.82(\lambda_3 + 2\lambda_4). \tag{18}$$

The thickness measured from 10 % to 90 % of full density is $2.98b$, to be equated with an experimental value of about 2.4×10^{-13} cm.

With the same approximations the surface energy is represented by the term

$$\tfrac{8}{5}(b/r_0)E_0A^{\frac{2}{3}}. \tag{19}$$

The coefficient of $A^{\frac{2}{3}}$ given by Cameron [12]) is 25.8 MeV, while with $b = 0.80 \times 10^{-13}$ cm this formula gives 19.3 MeV; the discrepancy is hardly significant in view of the approximations employed.

Finally we may consider the symmetry energy that arises when the numbers of protons and neutrons are unequal, expressed in the semi-empirical mass-formula by a term

$$E_{sy}(A-2Z)^2/A. \tag{20}$$

To calculate E_{sy} it is assumed that protons and neutrons form two separate Fermi gases with densities $(Z/A)\rho_0$ and $(A-Z/A)\rho_0$ respectively. The result gives

$$\begin{aligned} E_{sy}/E_0 = 1.38(&\tfrac{5}{9} - \tfrac{10}{9}\lambda_1 x_1 + \tfrac{8}{9}\lambda_2 + \tfrac{2}{5}\lambda_2 x_2) \\ &- \tfrac{1}{3}\lambda_0(1+2x_0) - \lambda_3 - 2\lambda_4. \end{aligned} \tag{21}$$

It should be noted that this formula bears no simple relation to the effective mass; also that it depends upon the exchange character of the interactions (that is upon the x_i), but fortunately not very sensitively. An estimate based upon the apparent strengths of singlet and triplet interactions in light nuclei, suggests that the contributions of these terms to the expression (21) is about 0.25. The elimination of λ_0 by (15) then gives

$$E_{sy}/E_0 = 2.55 + 1.23\,\lambda_2 - 1.17\,\lambda_3 - 2.67\,\lambda_4. \tag{22}$$

The value determined empirically by Cameron [12]) is $E_s = 31.5$ MeV; this rather large value arises because Cameron includes in his formula an additional surface symmetry energy term.

Corresponding symmetry energy terms arise also in the single-particle potential. Thus we obtain for *protons* a velocity-dependent potential (exclusive of the Coulomb term) described by an effective mass M_p^* and depth V_p, where

$$M/M_p^* = 1 + \lambda_1 + \lambda_2 + (N-Z/A)(\tfrac{1}{3}\lambda_1 - \tfrac{1}{5}\lambda_2) \tag{23}$$

and

$$V_p = V_0 + (N-Z/A)[-2E_{sy} + 1.38\tfrac{1}{9}E_0(10 + 5\lambda_1 + 13\lambda_2)]. \tag{24}$$

For *neutrons* similar equations hold with N and Z interchanged.

† Equation (33) of ref.[13]).

The term involving E_{sy} can be related to the Coulomb potential through the stability condition (which determines Z as a function of A); this shows that the term is equal to

$$-1.2(Ze^2/R) \tag{25}$$

the factor 1.2 arising from the *average* of the Coulomb potential within the nucleus. It may easily be verified that the other terms are such that the total energy of the topmost proton

$$V_p+2.30\,E_0(M/M_p^*)[1-\tfrac{2}{3}(N-Z/A)]+1.2(Ze^2/R) = -B. \tag{26}$$

4. Light Nuclei

The expectation value of the effective Hamiltonian for light nuclei has been calculated on the assumption that the shell-model wave-functions of these nuclei can be approximated by suitable oscillator wave-functions. The scale-constant of the wave-functions is described by the parameter y such that the exponential factor is $\exp[-\frac{1}{2}y^2(r/r_0)^2]$. The polynomial form assumed for $t(\mathbf{k}', \mathbf{k})$ leads then to an expression for the energy which is a polynomial in y; the value of y is determined by minimising the energy.

The contribution of the mean potential, described by the strength parameters $\lambda_0 \ldots \lambda_3$, to the total energy may be written

$$E/E_0 = Ay^2+B_0\lambda_0 4.255\,y^3+(B_1\lambda_1+B_2\lambda_2)12.766\,y^5+B_3\lambda_3 6.970\,y^6, \tag{27}$$

where the coefficients A, B_n depend upon the particular shell-model state considered; their values for a number of states (or differences of states) are given in table 1.

TABLE 1

State	A	B_0	B_1	B_2	B_3
s^3	3	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
s^4	$\frac{9}{2}$	1	1	0	1
s^4p-s^4	$\frac{5}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$
s^4p^{12}	$\frac{69}{2}$	$\frac{31}{4}$	$\frac{35}{4}$	4	$\frac{116}{9}$
$s^4p^{12}-s^4p^{11}$	$\frac{5}{2}$	$\frac{7}{8}$	$\frac{25}{24}$	$\frac{1}{2}$	$\frac{37}{18}$
$s^4p^{12}d-s^4p^{12}$	$\frac{7}{2}$	$\frac{9}{16}$	$\frac{35}{48}$	$\frac{13}{24}$	$\frac{38}{36}$
$s^4p^{12}2s-s^4p^{12}$	$\frac{7}{2}$	$\frac{21}{32}$	$\frac{91}{96}$	$\frac{47}{48}$	$\frac{31}{18}$
$s^4p^{12}d^{20}2s^4$	$\frac{237}{2}$	$\frac{1945}{64}$	$\frac{2625}{64}$	$\frac{1760}{64}$	$\frac{2060}{27}$

For other nuclei in the p-shell, which cannot be described by a single determinantal wave-function, we must determine the various two- (and three-)body matrix elements of the effective potential in states formed by two (or three) p-wave nucleons. These matrix elements may also be written

in the form (27); the values of B_n for the contributions of the mean, spin-independent potential are given in table 2.

TABLE 2

State	B_0	B_1	B_2	B_3
$p^2[2]S$	$\frac{5}{24}$	$\frac{3}{8}$	0	$\frac{5}{18}$
$p^2[2]D$	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{9}$
$p^{-2}[2]S$	$\frac{5}{24}$	$\frac{3}{8}$	0	$\frac{25}{27}$
$p^{-2}[2]D$	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{10}{27}$
p^2 or $p^{-2}[11]P$	0	0	$\frac{1}{15}$	0
$p^{\pm 3}[3]P$	0	0	0	$\pm\frac{7}{36}$
$p^{\pm 3}[3]F$	0	0	0	$\pm\frac{1}{18}$

The three-body contact interaction gives no contribution to states of (p^3) other than the most symmetric [3] states. The spin-dependent parts of the potential contribute proportionately; so that in a spin-triplet state B_0 should be multiplied by $(1+x_0)$, and so on.

Since the total energy should be stationary for the 'correct' value of y, it is not necessary to fix y self-consistently with great precision. In practice y has been fixed for each nucleus by choosing a particular set of λ_n, close to those finally adopted, and a particular low-lying shell-model to fix the B_n. Some of these values and the corresponding r.m.s. radii are given in table 3.

TABLE 3

Mass Number	y	Calc. r.m.s. radius 10^{-13} cm	Exp. r.m.s. radius
3	0.788	1.41	–
4	0.743	1.58	1.61
6	0.709	1.97	2.44
7	0.690	2.11	2.20
8	0.683	2.19	–
12	0.676	2.35	2.37
14	0.670	2.41	–
16	0.663	2.46	2.64
40	0.596	3.21	3.52

The experimental radius values are taken from Hofstadter [14]).

There is a tendency for our model to predict rather small radii; this might be expected on two counts: (a) oscillator wave-functions have less 'tail' than realistic ones, and (b) correlations in the real wave-function may increase the smearing of the surface region, by introducing higher single-particle states with larger r.m.s. radii.

5. Determination of Parameters

The first object of the present analysis has been to see whether a reasonable 'fit' to the experimental energies can be obtained with a mean potential depending only on the four parameters $\lambda_0 \ldots \lambda_3$. To simplify the comparison we have demanded that the conditions (15) should be satisfied exactly (with $\lambda_4 = 0$) and used these to express all calculated energies in terms of λ_2 and λ_3 only. Each comparison with experiment then determines a linear relation between these two parameters, which can conveniently be displayed graphically.

A number of these lines are shown in fig. 1. In the comparisons indicated the observed binding energies have been corrected for the Coulomb contribution, and for spin-orbit splitting, to give an 'observed' mean nuclear binding that can be compared directly with the calculations. The first point

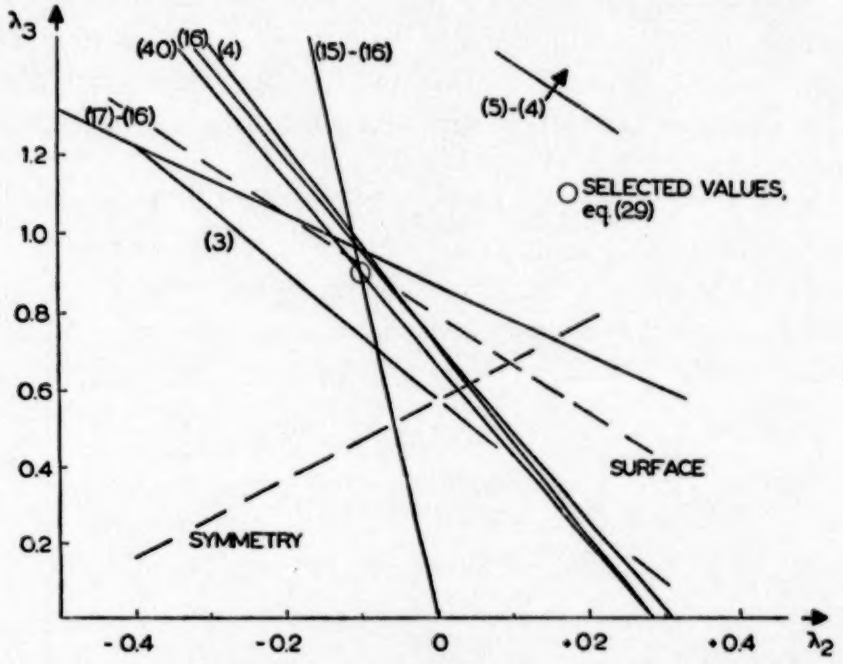

Fig. 1. Relations between the parameters λ_2 and λ_3 determined by the binding energies of some light nuclei (with mass numbers) and by the empirical surface and symmetry energies.

to note is that the total binding energies of the closed shell nuclei are consistent, and they indicate a relation between the parameters which has been taken as

$$\lambda_2 + 0.415\,\lambda_3 = 0.27. \tag{28}$$

The positions of the other lines suggest strongly that a positive value of λ_3, of the order of unity, is needed to obtain any sort of consistency among the other data. It would be pointless in this preliminary analysis to attempt to determine a 'best-possible' set of values, so a value of 0.9 was chosen for λ_3

by inspection fixing the other parameters by (16) and (28); these values are

$$\begin{aligned} \lambda_0 &= -4.155 & \lambda_3 &= 0.9 \\ \lambda_1 &= 0.723 & \lambda_2 &= -0.104 \end{aligned} \tag{29}$$

and the corresponding values of the t_n in the potential (6) are ($f = 10^{-13}$ cm)

$$\begin{aligned} t_0 &= -1072\, f^3 \text{ MeV} & t_3 &= +8027\, f^6 \text{ MeV} \\ t_1 &= +461\, f^5 \text{ MeV} & t_2 &= -40\, f^5 \text{ MeV}. \end{aligned} \tag{30}$$

The qualitative conclusions are that there is a considerable repulsive three-body force tending to reduce the strength of interaction between two nucleons in nuclear matter compared with free space; and that this effective interaction is predominantly of 'Serber' type, being very weak in odd states. The uncertainty in the determination of λ_3 is of the order of ± 0.2, and λ_2 is scarcely significantly different from zero. (See also section 7, below.)

With these values (29) we obtain the following results. The mean single-particle potential (eq. (17)) has

$$V_0 = -80 \text{ MeV}, \qquad M/M^* = 1.62, \tag{31}$$

the quotient of which is $M^*V_0/M = -49$, close to the values commonly used in optical model analyses. The formulae (23) and (24) indicate an appreciable difference in the values for protons and neutrons in heavy nuclei, but the changes are such that the product M^*V would be expected to remain nearly constant.

The surface thickness constant b, determined by eq. (18), comes out to be $0.76\, r_0 = 0.84 \times 10^{-13}$ cm.

The symmetry energy E_s, eq. (22), is 92 MeV.

For the light nuclei the calculated radii have been given in table 3; some calculated binding energies are given in table 4, and compared with the observed values, corrected for Coulomb and spin-orbit effects.

TABLE 4

Mass Number	Calculated B.E. (MeV)	'Corrected' Experimental B.E. (MeV)
3	11	8.5
4	27	29
5 relative to 4	−4	−1
15 relative to 16	−19	−19
16	138	140
17(d) relative to 16	1	1
17(2s) relative to 16	0	3
40	410	413

The calculation of the energies of other nuclei in the p-shell requires knowledge of the spin-orbit coupling and of the exchange nature of the interactions and lies outside the scope of the present paper.

6. Four-Body Interactions

The agreement obtained in the previous section by use of the very simple effective mean potential is satisfactory, but it is also necessary to examine how far the conclusions would be modified by including additional terms in the potential. As far as the many-body parts are concerned the simple three-body contact interaction (5) could be generalised both by including momentum-dependent terms (i.e. finite range) and by adding four- (or more) body interactions. The former possibility would require the introduction of several new parameters, and determination of significant values for them would be impractical at the present stage; we consider briefly below the effects of introducing a four-body interaction in the simplest possible way.

This interaction, taken in the zero-range form given by eq. (8), introduces just one new parameter t_4, or λ_4 defined by eq. (13). This term has already been included in the formulae of section 3; for the light nuclei described by oscillator wave-functions there will be an additional term in the expression for the energy (27), which may be written analogously as

$$B_4\lambda_4\, 13.622\, y^9. \tag{32}$$

The definition is chosen so that $B_4 = 1$ for mass 4 (i.e. for the state s^4); the value for the state s^4p^{12} is found to be $\frac{5089}{256}$. We have used the total binding energy of O^{16} to fix relation between the parameters in addition to eq. (15), analogous to eq. (28); this is

$$\lambda_2+0.415\,\lambda_3+0.680\,\lambda_4 = 0.27, \tag{33}$$

and with it all energies have been expressed in terms of λ_3 and λ_4 only.

In many of the formulae of section 3 these parameters occur in the combination $\lambda_3+2\lambda_4$, which may be understood in the following way. The three- and four-body terms may be represented by an equivalent density-dependent two-body contact interaction,

$$t'_0(\rho) = t_0+\tfrac{1}{6}\rho t_3+\tfrac{1}{96}\rho^2 t_4. \tag{34}$$

TABLE 5

Energy	X
Surface Energy	2.3
Symmetry Energy	2.1
Single-Particle	2.0
s^4	2.4
s^4p-s^4	1.6
$s^4p^{12}-s^4p^{11}$	1.6
$(p^2)S$	2.5
$(p^{-2})S$	2.0

The derivative t'_0 with respect to density at the equilibrium density is proportional to $t_3+\frac{1}{8}\rho_0 t_4$, that is to $\lambda_3+2\lambda_4$.

We have found that the combinations which occur are closely similar in all cases. Writing them as $\lambda_2+X\lambda_3$, values of X are given in table 5.

The scatter of values is such that a significant determination of λ_4 is impossible. The inference suggested is that only the first derivative of an equivalent density-dependent two-body interaction is important, or that the interaction at densities far from ρ_0 is relatively unimportant.

7. D-Wave Interactions

The shortcomings of the assumed form (6) for the effective two-body interaction may be divided into incorrect *energy* dependence (i.e. upon $\mathbf{k}^2$ and $\mathbf{k}'^2$) and *angular* dependence (i.e. upon $\mathbf{k}\cdot\mathbf{k}'$). Since the range of momenta is much the same in all nuclei it is possible that the former type of error may not seriously affect matrix elements; on the other hand the small angular momenta involved in light nuclei mean that scattering at angles far from 0 or π may be important. The form of (6) corresponds to scattering only in S or P states; the analysis of nucleon-nucleon scattering at similar relative momenta indicates considerable amounts of D-wave (see section 9); it is expected therefore that an important correction to (6) might be represented by a term

$$t_D[\mathbf{k}^2\mathbf{k}'^2-(\mathbf{k}\cdot\mathbf{k}')^2]. \tag{7}$$

We define a dimensionless parameter

$$\lambda_D = \tfrac{1}{4}(\rho_0/E_0 r_0^4)\tfrac{3}{4}t_D \tag{35}$$

and a coefficient B_D such that the contribution of (7) to the energy of a state built of oscillator wave-functions is

$$B_D\lambda_D\ 12.766\ y^7. \tag{36}$$

To the surface thickness parameter (7) contributes

$$\Delta(b^2/r_0^2) = \lambda_D\ 1.8\ r_0^2\rho^{\frac{2}{3}} \approx 0.72\ \lambda_D \tag{37}$$

on average. The values of B_D for some of the states listed in table 1 and 2 are given in table 6.

As with the four-body interaction we have eliminated a parameter by fitting exactly the binding energy of O^{16}, giving the relation

$$\lambda_2+0.415\ \lambda_3-0.716\ \lambda_D = 0.27. \tag{38}$$

It turns out that when λ_0, λ_1 and λ_2 are eliminated by (15) and (38), the surface energy and the energy of p^{-1} in O^{16} are almost independent of λ_D. As these two quantities were fitted well by the values in section 5, no change in the value of λ_3, nor therefore of λ_0, is indicated.

On the other hand there does appear to be a small gain in consistency by assuming a small positive value for λ_D. The calculations that have been made are not extensive enough to justify a definite choice, but for illustration we quote the following possible values of λ_1, λ_2 and λ_D,

$$\lambda_1 = 0.62, \qquad \lambda_2 = 0, \qquad \lambda_D = 0.15. \tag{39}$$

TABLE 6

State	B_D
s^4	1
$s^4p - s^4$	$\frac{1}{4}$
s^4p^{12}	$\frac{31}{4}$
$s^4p^{12} - s^4p^{11}$	$\frac{7}{8}$
$(p^2)S$	$\frac{55}{72}$
$(p^2)D$	$-\frac{1}{36}$

8. The Three-Body Potential

The preceding analysis suggests the need for an appreciable three-body interaction in calculations using shell-model wave-functions. We must now ask how far this conclusion is compatible with the general success of such calculations in which two-body forces alone have been used.

In the p-shell intermediate coupling calculations, such as have been extensively made by Kurath [15]), depend only upon three parameters L, K and a but even these are not unambiguously given by the comparison of calculated and experimental energy levels. The principal effect of the three-body interaction would be to *decrease* the central-force integrals towards the end of the shell, (apart from any effects arising from changes in the single-partical orbits). There is indeed some indication of this, but its significance cannot be assessed so long as the single-particle spin-orbit is regarded as an arbitrary parameter. In three-particle calculations, such as those of Elliott and Flowers [16]), for mass 19, the effect of the three-body interaction will be to raise the levels of states of maximum orbital symmetry by a few tenths of a MeV; since the low-lying levels are mainly composed of such states, the additional interaction is unlikely to cause great changes in the level order predicted.

In their analysis of interactions in mass 40, Pandya and French [17]) have found some evidence for a three-body contact interaction precisely of the form that we have postulated. They find for its strength $t_3 = 28\,000\ f^6$ MeV, which is several times *larger* than our preferred value (about $8\,000\ f^6$ MeV); we would ascribe some part of this difference to the fact that Pandya and

French used an oscillator constant $r_t = 3f$, larger than the value derived from table 3, which is $r_0\sqrt{2}/y = 2.64f$; the scaling factor $(2.64/3.00)^6$ would reduce the value of t_3 needed to give the same values for the matrix elements to 13 000 f^6 MeV.

The results of Talmi and Thieberger [18]) on the binding energies of shells in the j-j coupling model appear, on the other hand, to indicate that two-body interactions alone give a good account of the variation of binding energy within a shell. However, the fact that cubic terms in the energy are small does not necessarily imply that three-body interactions are absent, because the latter may be marked by changes (e.g. in the single-particle wave-functions) which have been ignored in the empirical argument. To illustrate this point suppose that the 'core' is kept fixed, but that the orbit considered can be described by a variable scale parameter y, such for example as was used in section 4; the energy of a configuration j^n will contain terms such as

$$nA(y)+\tfrac{1}{2}n(n-1)B(y)+\tfrac{1}{6}n(n-1)(n-2)C(y) \qquad (40)$$

where A, B and C are the single-particle energy, the mean binding energy between a pair, and that between three particles. For $n = 1$, y will have the value y_0 that makes A a minimum, and we can write

$$\begin{aligned} A(y) &= A_0+\tfrac{1}{2}A_2(y-y_0)^2+\ldots, \\ B(y) &= B_0+\tfrac{1}{2}B_1(y-y_0)+\ldots. \end{aligned} \qquad (41)$$

We substitute these in (40) and minimise with respect to y; the terms in the energy up to order n^3 are then

$$nA_0+\tfrac{1}{2}n(n-1)[B_0-\tfrac{1}{4}B_1^2/A_2]+\tfrac{1}{6}n(n-1)(n-2)[C_0-\tfrac{3}{4}B_1^2/A_2]. \qquad (42)$$

Since A_2 is positive the correction term tends to decrease the real three-body interaction measured by C_0. In the p-shell typical mean values taken from our calculations are

$$y_0^2A_2 = 79 \text{ MeV}, \qquad y_0B_1 = -3.4 \text{ MeV}, \qquad C_0 = 0.18 \text{ MeV}$$

for which $\frac{3}{4}B_1^2/A_2 = 0.21$ MeV, reducing considerably the apparent three-body interaction.

In another direction some confirmation of our estimate for the strength of the three-body interaction is found in the value of the 'rearrangement energy' arising in the many-body theory [19]). This is the difference between the real binding energy of the topmost particle in the nucleus (equal, for infinite nuclear matter, to B) and that given by the topmost energy level in the self-consistent single-particle potential as conventionally defined for pairwise interactions. In our analysis, with the explicit introduction of many-body potentials, these two quantities are the same, and Squires [9]) has shown that the rearrangement energy is simply related to the contribution of the many-body potentials to the binding energy. Comparison of

eqs. (16) and (12) shows that this 'rearrangement energy' must be equal to

$$E_0[(2\lambda_0+3\lambda_3+4\lambda_4)-2(\lambda_0+\lambda_3+\lambda_4)] = E_0(\lambda_3+2\lambda_4) \qquad (43)$$

again depending only on the combination $\lambda_3+2\lambda_4$ as we should expect.

Our estimate for this quantity is about 15 MeV (with an uncertainty of the order of 20 %), to be compared with the value of 17 MeV found in the numerical calculations of the many-body theory.

9. The Two-Body Potential

Our analysis of the effective potential has yielded

(a) A representation of the mean (i.e. average of singlet and triplet) even-state potential in the form

$$\delta(\mathbf{r}_1-\mathbf{r}_2)[t_0+\tfrac{1}{2}t_1(\mathbf{k}^2+\mathbf{k}'^2)+t_3\bar{\rho}] \qquad (44)$$

where $\bar{\rho}$ is an average of the nuclear density at the point of interaction.

(b) The conclusion that the odd-state average interaction (which is 90 % the triplet-triplet interaction) is relatively weak, possibly slightly attractive. We can compare the matrix elements calculated with (44) with the corresponding averages used in shell-model calculations.

At the beginning and end of the p-shell we find the values given in table 7 for the average S and D matrix elements.

TABLE 7

Matrix Element	Energy (MeV)
$(p^2)\overline{S}$	−7.9
$(p^2)\overline{D}$	−5.0
$(p^{-2})\overline{S}$	−2.0
$(p^{-2})\overline{D}$	−2.1

The values at the beginning are in accord with those commonly assumed; taking the conventional singlet-triplet ratio of 0.6, they correspond to $L/K = 6$, $K = 1.25$ MeV. At the end of the shell however, the repulsive effects of the three-body interaction have greatly reduced their strength. The effect has probably been exaggerated by the simple forms that we have assumed, but is not necessarily in violent contradiction with observation because the level scheme is dominated by the spin-orbit coupling towards the end of the shell †.

There is very little evidence on the odd-state interaction from level structure. In the calculations of Kurath [15]) an exchange mixture was used in which the triplet-odd interaction is strongly repulsive, but the effect on the low-lying levels which are observed is generally small. In his analysis of the

† See, for example fig. 2 of ref. [7]). The disagreement for mass 13 in fig. 3 is due, in part at least, to incorrect assumptions about the tensor force.

structure of Li^6, Soper [20]) has found the 3.3 interaction to be weakly attractive, but the 1.1 to be undetermined. Huber [21]) has made a systematic analysis of the p-shell with two-body forces and also finds a weak attraction in the mean in odd states.

Beyond the p-shell so many different matrix elements are involved in the calculations that this type of comparison is not helpful.

It is also interesting to compare our results with a simple estimate based on the phase-shift analysis of nucleon-nucleon scattering. For this we assume that

(a) $t(\mathbf{k}', \mathbf{k})$ is proportional to the scattering amplitude from relative momentum $\mathbf{k}$ to $\mathbf{k}'$ with every term $e^{i\delta} \sin \delta$ replaced by δ,

(b) the phases δ are the same as for free-particle scattering,

and when only S and P waves are included in our approximation for t, also that

(c) the comparison is made only for forward and backward scattering, $\mathbf{k} = +\mathbf{k}'$.

These assumptions lead to the identifications

$$\begin{aligned}(\rho_0/E_0)(t_0+t_1k^2) &= -(6/kr_0)(\bar{\delta}_0+5\bar{\delta}_2+9\bar{\delta}_4+\ldots)\\ (\rho_0/E_0x_0^2)t_2k^2 &= -(6/kr_0)(3\bar{\delta}_1+7\bar{\delta}_3+\ldots)\end{aligned} \tag{45}$$

where the even-state averages are of 50 % triplet and 50 % singlet, and the odd state of 90 % triplet and 10 % singlet, and the average over J for the triplet states has the usual $(2J+L)$ weighting. The laboratory energy of scattering is

$$E = 67(kr_0)^2\text{MeV}. \tag{46}$$

The numerical comparison is illustrated in fig. 2. The solid lines represent the left-hand sides of (45) as a function of energy with the values of the t_n given by (30). The points marked represent values of the right-hand sides computed from the phase-shifts given by Gammel and Thaler [3]) or by Marshak and Signell [4]). The agreement is quite good in view of the limitations of the assumed form for $t(\mathbf{k}', \mathbf{k})$.

In the discussion of the symmetry energy, eq. (21), we required an estimate of the exchange dependence of the two-body potential. The odd-state exchange dependence, specified by x_2, cannot easily be determined and not at all without discussion of the non-central tensor force components of the potential. In even states x_0 and x_1 determine the relative strengths of triplet and singlet interactions; in most cases they occur in a similar combination (approximately $x_0-0.5x_1$), and if, arbitrarily, we put $x_1 = 0$ inspection of the level structure in masses 6 and 14 suggests a value of x_0 around 0.1. This can be compared with the phase-shifts by making the identification

$$(\rho_0/E_0)(x_0t_0+x_1t_1k^2) = -(6/kr_0)(\tilde{\delta}_0+5\tilde{\delta}_2+\ldots). \tag{47}$$

In this formula $\tilde{\delta}$ denotes half the difference of triplet and singlet phases. A good fit to the phase-shifts is given by $x_0 = x_1 = 0.12$ approximately.

For the D-wave contribution to the potential, defined by (7), the phase-shift identification is

$$(\rho_0/E_0)t_D k^4 = (9/kr_0)5\tilde{\delta}_2 \tag{48}$$

or equivalently

$$\lambda_D = \tfrac{27}{16}5\tilde{\delta}_2/(kr_0)^5. \tag{49}$$

The phase-shifts at 100 MeV then give a value around 0.3 for λ_D; this is larger by a factor 2 than the value previously suggested, eq. (39).

If the D-wave contribution were increased to this value the empirical analysis would then determine a compensatingly smaller value for t_1 which would make the agreement in eq. (45) less satisfactory. This however, might be compensated by momentum-dependent terms in the three-body interaction which we have ignored for the sake of simplicity.

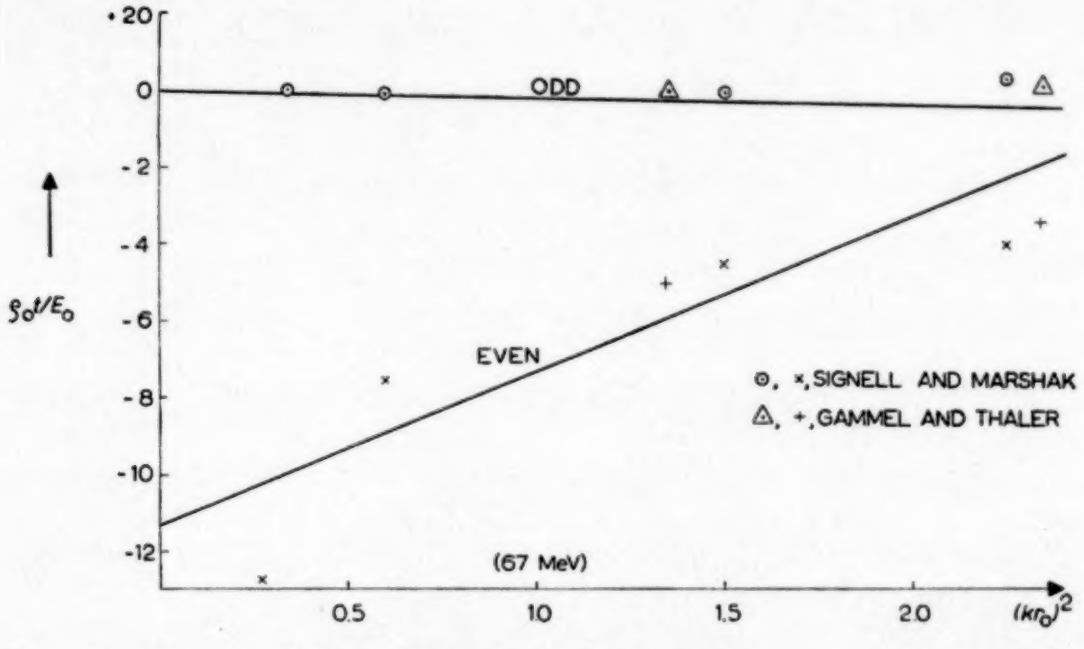

Fig. 2. The mean central potential in even and odd states compared with the estimate based on scattering phase-shifts.

10. Conclusion

We have shown that, within the limitations imposed by the simple forms assumed for ease of computation, a consistent account of nuclear binding energies requires the introduction of many-body terms, or that if the interaction is forced into a two-body form the latter must be allowed to be density dependent (at least); this conclusion is in agreement with the results of numerical calculations made on the many-body problem. Our empirical

argument ultimately rests upon the fact the two- and three-body interactions contribute in different relative strengths to the *total* binding energy and to the interaction between two given particles or to the self-consistent single-particle potential; this, in effect, removes the discrepancy that has been known for a long time between the strengths of two-body interaction needed to explain level structure and to explain total energies.

We have incidentally constructed a simple model for the effective interaction, but we must emphasise that it can only reasonably be used in Born approximation where high momentum components are irrelevant. The Serber character of the interaction is in agreement with evidence provided by nucleon scattering off light nuclei.

The empirical elucidation of the spin-dependent parts of the interaction is much more difficult. This is because we must now consider nuclei removed from the closed shells, and the appropriate 'model' wave-functions are less certain on account of possible collective long-range correlations. These are present even in the p-shell, so that it is uncertain how far the empirical energy matrices used in shell-model calculations reflect the effective interaction without modifications due to the neglect of distortion in the model wave-function.

The most important problem is to relate the effective interaction to some 'real' interaction between the particles. Current formulations of the many-body problem emphasise the self-consistent determination of a two-body interaction; while this may be a suitable approach for the idealised problem of 'infinite nuclear matter', we suggest that it may be more profitable, for the application of the theory to finite nuclei, to leave explicitly the many-body effects as additional terms in the energy, so as to give a Hamiltonian that is independent of the particular state considered.

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THE SPIN-ORBIT INTERACTION IN NUCLEI

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Abstract: The analysis previously made of the average nuclear potential has been extended to consideration of the spin-orbit interactions. It has not been possible to find a satisfactory two-body interaction consistent with all the data; that suggested by the phase-shift analysis of nucleon-nucleon scattering is just within the region of possible forms.

1. Introduction

In a previous paper [1]) we have examined empirically the nature of the *average* interaction between nucleons which is effective in the shell-model description of a nucleus. The next logical step in the determination of the effective interaction is a discussion of the spin-orbit coupling terms, since these are already manifested for a single particle outside a closed shell. This paper contains an account of our analysis of the data made in a way similar to that used for the mean potential, and incidentally brings up to date an earlier investigation of the problem [2]).

The problem of spin-orbit coupling is distinguished from that of the average potential by the comparative paucity of data, and the sensitive dependence of their interpretation upon details of the wave-functions assumed. In our first investigations we had assumed that the spin-orbit interaction might be described by a two-body interaction of very short range, written in the notation of ref. [1]) as

$$\delta(\mathbf{r}_1-\mathbf{r}_2)i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}'\times\mathbf{k}V \qquad (1)$$

with some constant V.

This simple hypothesis cannot be made to fit the facts adequately. It fails

(a) to describe the strong variation of coupling through the p-shell,

(b) to account for the different couplings in masses 15 and 17,

(c) to explain the large amount of coupling indicated by the calculations on the order of single-particle levels.

It can be modified either by allowing V to be dependent upon the momenta $\mathbf{k}'$ and $\mathbf{k}$, or by allowing it to depend upon density through the introduction of similar three- (or many-)body terms. The possible forms of three-body

interaction are numerous and have not been explored, but as trial calculations have not indicated any obvious gain in consistency through their inclusion they will not be discussed further in this paper.

We have considered the three-parameter form of V that was introduced in ref.[2]), allowing linear dependence upon $\mathbf{k}^2$ and upon $\mathbf{k}\cdot\mathbf{k}'$. The former dependence does not appear to be significant and cannot be determined empirically; on the other hand the introduction of 'angular' dependence in V leads to greater consistency.

The formulae needed for this discussion are collected in the following two sections. The possible determination of the parameters is discussed in section 4, together with an estimate based upon free-nucleon phase shifts as in ref.[2]); those numerical values were founded on the phase-shift analysis of Feshbach and Lomon [3]), and have been replaced by values derived from the analyses of Gammel and Thaler [4]) and of Marshak and Signell [5]).

2. The Two-Body Interaction

The discussion of $V(\mathbf{k}', \mathbf{k})$ will be based on the three-parameter form

$$V = v_0+\tfrac{1}{2}v_1(\mathbf{k}^2+\mathbf{k}'^2)+v_2\mathbf{k}'\cdot\mathbf{k}. \tag{2}$$

It is convenient also to introduce the dimensionless parameters

$$\begin{aligned}\mu_0 &= (\rho_0/E_0r_0^2)\tfrac{3}{2}v_0\\ \mu_1 &= (\rho_0/E_0r_0^4)\tfrac{3}{2}v_1\\ \mu_2 &= (\rho_0/E_0r_0^4)\tfrac{1}{2}v_2;\end{aligned} \tag{3}$$

$\rho_0 = \frac{4}{3}\pi r_0^3$ is the density of nuclear matter, and $E_0 = 1/(2Mr_0^2)$ as in ref.[1]).

In the p-shell two distinct matrix elements only of the spin-orbit interaction are needed. In the notation of Elliott [6]) they have the following values for oscillator wave-functions (characterised by an exponential factor $\exp[-\frac{1}{2}(yr/r_0)^2]$):

$$\begin{aligned}D = E &= -0.177\ y^5(\mu_0+2.5\ y^2\mu_1)E_0,\\ G &= -0.532\ y^5\mu_2E_0.\end{aligned} \tag{4}$$

The single-particle spin-orbit coupling is usually written as $a\ \mathbf{l}\cdot\mathbf{s}$. In terms of the matrix elements E and G, its values at the beginning and end of the p-shell are

$$\begin{aligned}&\text{p};\qquad a = 6E\\ &\text{p}^{-1}\text{:}\qquad a = -9E-5G.\end{aligned} \tag{5}$$

In mass 6, the splitting of the three ^{3}D levels (which is the most obvious manifestation of the spin-orbit coupling (Soper [7])) is determined by

$$a_{\text{eff}} = 6E+2G \tag{6}$$

apart from effects of the tensor forces.

According to eq. (5) the value for mass 15 is

$$a = 1.60\, y^5(\mu_0+2.5\, y^2\mu_1+1.67\, y^2\mu_2) \tag{7}$$

while in mass 17 a straightforward calculation gives for the d-level

$$a = 1.33\, y^5(\mu_0+2.9\, y^2\mu_1+1.2\, y^2\mu_2). \tag{8}$$

3. The One-Body Interaction

The effective single-body spin-orbit interaction corresponding to the form (2) for $V(\mathbf{k}', \mathbf{k})$ was worked out by Bell and Skyrme [2]) for the case of a shell-model wave-function with four nucleons in each orbital state and a spherically symmetric density distribution. The result could be written in the form of an interaction

$$A(r)\mathbf{l}\cdot\mathbf{s} \tag{9}$$

where

$$\begin{aligned} A(r)/E_0 = {} & \mu_0(r_0^2/\rho_0)r^{-1}\mathrm{d}\rho/\mathrm{d}r \\ & + \tfrac{1}{4}(\mu_2-\mu_1)(r_0^4/\rho_0)r^{-1}(\mathrm{d}/\mathrm{d}r)(\nabla^2\rho) \\ & + \tfrac{1}{4}(\mu_2+\mu_1)(r_0^4/\rho_0)(\tfrac{5}{3}r^{-1}\mathrm{d}\tau/\mathrm{d}r-r^{-1}(\mathrm{d}\rho/\mathrm{d}r)\nabla^2). \end{aligned} \tag{10}$$

In this expression ρ is the nucleon density, τ is the density of (momentum) [2]) and an anisotropic term in the momentum density has been dropped.

In a large nucleus the contribution of the terms involving $r^{-1}(\mathrm{d}/\mathrm{d}r)(\nabla^2\rho)$ will be assumed to be small †. In the last group of terms we make an estimate by using the Thomas-Fermi approximation; for a particle at the top of the Fermi distribution $\nabla^2\rho = -\mathrm{d}\tau/\mathrm{d}\rho$, so the significant terms in (10) can be written in the form

$$(r_0^2/\rho_0)(r^{-1}\mathrm{d}\rho/\mathrm{d}r)[\mu_0+(\mu_1+\mu_2)\tfrac{2}{3}(\mathrm{d}\tau/\mathrm{d}\rho)r_0^2]. \tag{11}$$

From the T-F approximation $\frac{2}{3}r_0^2(\mathrm{d}\tau/\mathrm{d}\rho) = 1.5(\rho/\rho_0)^{\frac{3}{2}}$; with the surface distribution used in ref.[1]), the average value of $(\rho/\rho_0)^{\frac{3}{2}}$ is $\frac{3}{4}$. Finally therefore the combination of parameters that enters into the magnitude of the single-particle coupling in a large 'average' nucleus is

$$\bar{\mu} = \mu_0+1.12(\mu_1+\mu_2). \tag{12}$$

In comparing this result with the experimentally observed coupling in heavy nuclei it must be remembered that

(i) This formula can only be expected to be approximately correct for the average of neutron and proton spin-orbit couplings.

(ii) There may be other contributions to the single-body coupling arising from central and tensor forces, because the single-particle states are already in j-j coupling.

† This is hard to justify, but is supported by the manner in which the lines of the figure approach the 'infinite limit' as the nuclei become more tightly bound.

The term involving $(\mathrm{d}/\mathrm{d}r)(\nabla^2\rho)$ may be interpreted in terms of a finite range of the interaction, by writing

$$\begin{aligned}\bar{\mu}\rho(\mathbf{r})+\tfrac{1}{4}(\mu_2-\mu_1)r_0^2\nabla^2\rho \\ = \text{Average}\,\rho(\mathbf{r}+\mathbf{s}),\end{aligned} \tag{13}$$

in which the r.m.s. value of the range is given by

$$\text{Average}\ \mathbf{s}^2 = \tfrac{3}{2}r_0^2(\mu_2-\mu_1)/\bar{\mu}. \tag{14}$$

4. Determination of Parameters

From the light nuclei of masses 5, 6, 15 and 17 four relations between the parameters μ_n are obtained, given by eqs. (4) to (8) above when equated to the experimental splittings. In all of these the ratio of the coefficients of μ_1 and $\bar{\mu}_0$ varies only between 1.1 and 1.25, and the same is true of the large nucleus combination $\bar{\mu}$ given by eq. (12). It is not therefore possible to determine μ_0 and μ_1 separately, but only a combination which is approximately

$$\mu_0+1.2\,\mu_1 = \bar{\mu}_0; \tag{15}$$

this is equivalent to saying that the only important relative momenta are those near $(kr_0)^2 = 1.2$. In using the formulae we have accordingly replaced μ_0 by μ_0, and left out the μ_1 terms.

The evidence concerning $\bar{\mu}$, the effective coupling strength in heavy nuclei, is confusing. The only direct evidence available on doublet splittings is around Pb^{208}. These data have been analysed in terms of a short range interaction by Blin-Stoyle [8]), who found that they were consistent with a strength $\bar{\mu} = 0.44$, in the present notation.

On the other hand a number of calculations have been made of single-particle levels in a realistic well with spin-orbit coupling, and these indicate that a much larger value is needed to obtain the right level sequence. We have, in particular, examined the calculations of Ross, Lawson and Mark [9]); these authors took

$$A(r) = -\tfrac{1}{2}\lambda(\hbar/Mc)^2 r^{-1}\mathrm{d}V/\mathrm{d}r$$

with

$$V(r) = -V_0/[1+\exp\alpha(r-\mathrm{a})] \tag{16}$$

as the central potential; λ was an adjustable parameter. Calculations were made with and without an effective mass correction, and we have interpolated between them to the value $M/M^* = 1.62$, favoured in ref. [1]).

We have estimated the relation between $\mathrm{d}V/\mathrm{d}r$ and $\mathrm{d}\rho/\mathrm{d}r$ in two ways: on the basis of the Thomas-Fermi approximation and by direct numerical comparison with the surface shape derived in ref. [1]). They agree in giving, approximately

$$dV/dr = -0.8V_0\rho_0^{-1}\,d\rho/dr \tag{17}$$

which then leads to the identification

$$\bar{\mu} = \tfrac{1}{2}0.8(\hbar/Mcr_0)^2\lambda V_0/E_0 = 0.0144\,\lambda V_0/E_0. \tag{18}$$

The interpolated numerical value is then $\bar{\mu} = 1.8$, much larger than Blin-Stoyle's estimate.

Part at least of this discrepancy appears to be due to the fact that the calculations of Ross, Mark and Lawson give excessively large doublet splittings at the top of the well, e.g. the 1i splitting is some three times that estimated (from experiment) by Blin-Stoyle, while part may be due to the approximations used by the latter.

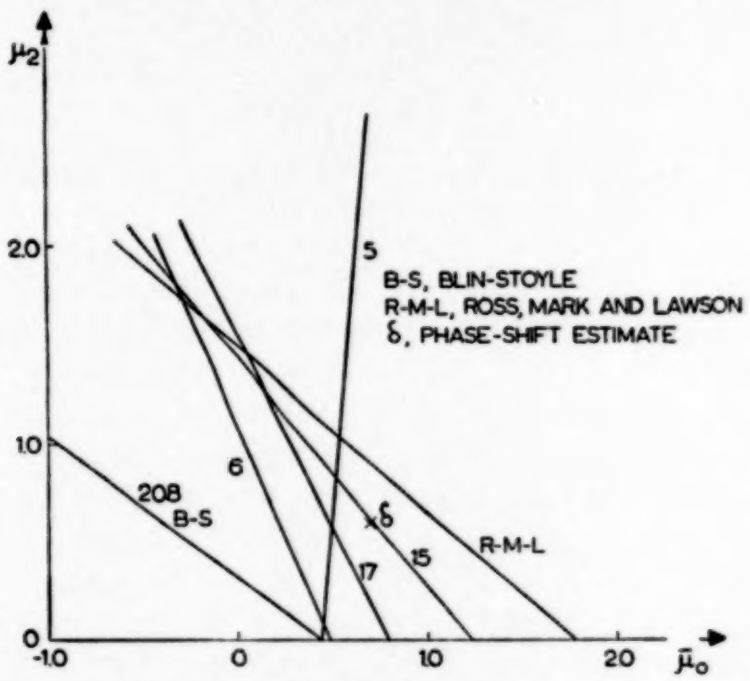

Fig. 1. Relations between $\bar{\mu}_0$ and μ_2 determined by doublet splittings in the mass numbers indicated, by single-particle level order and by the phase-shift estimate.

The various relations obtained between $\bar{\mu}_0$ and μ_2 are shown in the figure; the lines for masses 5 and 6 should be allowed considerable uncertainty on account of the greater inaccuracy of the assumed wave-functions for these lightly bound systems. There is no well-defined best pair of values. At the one extreme (disregarding mass 5 and Blin-Stoyle's value) a small value of $\bar{\mu}_0$ and large μ_2 are possible, with the interaction occurring principally in even states ($T = 0$); on the other hand the data may not be impossible to reconcile with the point marked δ which is that estimated on the basis of the nucleon-nucleon scattering phases.

This estimate has been made in the way described in refs.[1,2]), using δ not $\tan\delta$ as a measure of the energy shift; the formulae relevant here are

$$\bar{\mu}_0 = \tfrac{9}{8}(kr_0)^{-3}(5\delta_1{}^2 - 2\delta_1{}^0 - 3\delta_1{}^1),$$
$$\mu_2 = \tfrac{3}{8}(kr_0)^{-5}(14\delta_2{}^3 - 9\delta_2{}^1 - 5\delta_2{}^2). \qquad (19)$$

For $(kr_0)^2 = 1.2$, the value suggested by the ratio of the coefficients of μ_1 and μ_0, we obtain from the phase-shifts given by Marshak and Signell [5])

$$\bar{\mu}_0 = 0.7, \qquad \mu_2 = 0.5.$$

5. Conclusions

The analysis made in this paper does not unfortunately lead to such a clear conclusion as our analysis of the *mean* potential. Lane and Elliott [10]) found preference for a neutral interaction over a symmetrical one, but did not explicitly consider mixtures; our results do not disagree with theirs. Abraham [11]) who analysed the Li^7 splitting and compared requirements in this case with those for mass 5 preferred the exchange character $(1-P^r)$, i.e. interaction only in even states. On the other hand, Huper [12]) in an analysis of structure in the p-shell prefers a $(1+P^r)$ dependence.

These differences, together with the discrepancies between estimates of the coupling in heavy nuclei, make the determination of the effective interaction difficult. There is a need for extensive numerical calculations with realistic wave-functions.

It is not possible yet to say whether or not the two-body interaction will provide a sufficient account of all the phenomena. Jancovici [13]) has estimated the contribution of tensor forces to the reaction matrix, and finds that they are inadequate. The 'real' spin-orbit interaction between two nucleons may then account for most of the effective interaction investigated here, or it may be that many-body effects are relatively important: in the latter case the analysis made in this paper would have no significance.

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SOME DISTORTION EFFECTS IN THE NUCLEAR p-SHELL

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Abstract: Some reasons are given for disliking the tensor-force explanation of the long lifetime of C^{14}. It is suggested that the composition of the ground state of N^{14} is influenced by distortion in such a way as to simulate the presence of a tensor interaction. Calculations made with a particularly simple form of unitary distortion operator give an effect of the correct sign and magnitude.

1. Introduction

In two previous papers [1,2]) we have discussed the empirical evidence concerning the effective nuclear potential as far as the 'average' potential and the spin-orbit contribution were concerned. The programme was to be completed by a discussion of the spin-dependence of the central forces and of the tensor forces, for which it is necessary to consider the properties of nuclei with at least two particles (or holes) outside closed shells. Only in the p-shell (and only for some cases there) have the full energy matrices involving the non-central forces been evaluated, so that any detailed discussion of the tensor potential has to come from evidence within the p-shell.

In our first investigations [3]) we had relied heavily upon the one piece of evidence that seemed clearly to indicate a tensor force, arising from the analysis of the mass 14 system. Elliott [4]) and Visscher and Ferrell [5]) have shown that the long lifetime of C^{14}, and the level structure of N^{14}, can be understood within the framework of the $(s^4 p^{10})$ configuration provided there is a sufficient tensor coupling between the ^3S and ^3D components of the ground state of N^{14}. The value of this matrix element (around -2 MeV) then fixed the magnitude of the parameter T of the effective potential as represented in eq. (6) of ref.[1]).

We have found, however, that this interpretation is unsatisfactory on several counts:

(a) Calculations of the level structure of mass 13 give unacceptably small spacings. This is illustrated in fig. 3 of ref.[3]), and has been confirmed by further calculations with different parameters. The examination of the energy matrices suggests that the tensor force may largely be responsible for this effect; in the limit of j-j coupling the tensor force acts in direct opposition to the spin-orbit potential.

(b) A similar matrix element arising in the description of the Li^6 ground state would give rise to a quadrupole moment of the order of 7 mb, greatly in excess of the experimental upper limit of 1 mb; furthermore this matrix element would increase the splitting between the two lowest $J = 1$ levels, and the actual spectrum would be hard to explain.

(c) The numerical value required for the matrix element is rather large. A simple comparison with the nucleon-nucleon phase shifts, based on the same assumptions as in ref.[1]), gives for this element the value

$$-[6.25(\delta'_0-\delta'_2)\sin\varepsilon_1]/k^3 \text{ MeV}.$$

This is derived for oscillator wave-functions, with the size parameter determined in ref.[1]). The phase shift analysis of Gammel and Thaler [6]), at 90 MeV, leads to a value of about -0.5 MeV.

The conjunction of these difficulties (all of which separately might admit alternative explanations) has led us to look for an alternative interpretation of the mass 14 phenomena. This must of course lie in configuration admixture, as originally suggested by Inglis [7]). Some admixture will arise from the short-range correlations between pairs of particles, but according to the general philosophy of the shell-model these should be neglected when an effective potential is used to describe the interaction. On the other hand long-range correlations should be included explicitly in the shell-model wave-function. The most familiar of these are the distortion effects, and there is evidence for them even in the p-shell from the enhancement of E2 transition rates [8,9]).

In so far as distortion may provide a simple description of the state within the lowest configuration, it is only an alternative to the normal shell-model calculations; we are here interested explicitly in the asymmetrical nature of the single-particle orbits, which is equivalent to some configuration admixture. The usual phenomenological approach using wave-functions of the type discussed by Nilsson [10]) is possibly not very appropriate in the p-shell where only small or moderate distortions would be expected; moreover their consistent use requires that they be regarded as the generating functions of the states rather than the actual states, and this leads to great algebraic complexity.

We have therefore sought ways of expressing the deformation in terms of a unitary transformation of the wave-function, without any introduction of redundant co-ordinates. Then we can conveniently carry out calculations by inverting the transformation onto the Hamiltonian, giving a new 'effective' Hamiltonian, involving the distortion parameter, which is discussed in the usual way within the framework of the lowest configuration. The distortion parameter is determined by minimising the energy.

In this paper we have considered in detail the consequence of one particu-

lar way of describing the distortion. It is probably not the best or most reasonable form, but it leads to manageable calculations, and in so far as the variational principle is applicable it is one possibility that should logically be considered. We find that the ground state of N^{14}, which is predominantly $^3D_1$, tends to assume a prolate distortion, and that the transformed Hamiltonian contains an off-diagonal matrix element coupling ^3S and ^3D of the sign and approximate magnitude postulated in the p-shell analyses [4,5]). On the other hand in Li^6, where the ground state is predominantly $^3S_1$, there is no tendency to distortion and the off-diagonal matrix element is not generated.

2. The Distortion Operator

The volume-preserving operators describing infinitesimal distortions form a second rank tensor $D_q^{(2)}$, from which we shall generally drop the superscript. We adopt such a normalisation that, explicitly

$$D_0 = \sum(2zp_z - xp_x - yp_y). \tag{1}$$

It is convenient also to introduce the allied tensors $X_q^{(2)}$, $P_q^{(2)}$ with

$$\begin{aligned} P_0 &= \sum(2p_z^2 - p_x^2 - p_y^2) \\ X_0 &= \sum(2z^2 - x^2 - y^2). \end{aligned} \tag{2}$$

The combination

$$Q_q = \tfrac{1}{2}(P_q + X_q) \tag{3}$$

is the one introduced by Elliott [11]) in his study of the classification of oscillator wave-functions associated with the Hamiltonian

$$H_0 = \sum(\mathbf{r}^2 + \mathbf{p}^2) \tag{4}$$

where the characteristic unit of length b has been equated to unity.

We are looking for unitary transformations of the type

$$\exp(-i\alpha K) \tag{5}$$

in which α is a variable parameter, and K is a rotational scalar describing distortion along an intrinsic axis. In view of Elliott's work a natural choice for K would be the product of D and Q; the difficulty, however, is that Q has simple properties (i.e. commutation) only with the oscillator Hamiltonian (4). With a realistic Hamiltonian Q may commute in some average sense if the scalelength b is suitably chosen, but this is not at all evident.

In this paper we consider the consequences of the choice

$$K = \tfrac{1}{4}\,\mathbf{D}\cdot\mathbf{J} = \tfrac{1}{4}\sum_q (-1)^q D_q J_{-q}^{(2)} \tag{6}$$

where $J^{(2)}$ is the second rank tensor formed from the total angular momentum vector $\mathbf{J}$, normalised so that

$$J_0^{(2)} = 2J_z^2 - J_x^2 - J_y^2 = 3M^2 - J(J+1). \tag{7}$$

We shall write the 'real' wave-function ψ in terms of the 'model' ψ_0, lying within the lowest configuration, as

$$\psi = \exp(-i\alpha K)\psi_0. \tag{8}$$

This choice has the advantage that J commutes with the Hamiltonian; on the other hand it produces no effects in states of spin 0 or $\frac{1}{2}$. It is effective in the states of spin 1 which are the cause of this investigation; it is possible that the two transformations may produce equivalent effects, and in any case the transformation with K defined by eq. (6) has a right to be considered on its own account.

3. The Quadrupole Operator

The significance of the transformation is made clearer by consideration of the quadrupole operator. Apart from a factor e this is the tensor E_q with component

$$E_0 = \sum_{\text{protons}} (2z^2 - x^2 - y^2). \tag{9}$$

The expectation value of E_0 in the state ψ is the same as that of

$$E'_0 = \exp(i\alpha K) E_0 \exp(-i\alpha K) = E_0 + \Delta E_0 \tag{10}$$

in the state ψ_0.

To the first order in α,

$$\Delta E_0 = i\alpha[K, E_0]. \tag{11}$$

The most important term in this commutator is one arising from the commutation of D_0 with E_0 which has spherical symmetry, namely

$$\alpha J_0^{(2)}\Big(\sum_{\text{protons}} \mathbf{r}^2\Big). \tag{12}$$

The diagonal matrix element for the state with $M = J$ is

$$\alpha J(2J-1) \sum_{\text{protons}} \mathbf{r}^2. \tag{13}$$

We can compare this with the corresponding formula of the Bohr-Mottelson collective model (see, for example, Blin-Stoyle [12]) and obtain the equivalent intrinsic deformation

$$\beta = (2\alpha/5^{\frac{1}{2}})(J+\tfrac{1}{2})(J+\tfrac{3}{2}). \tag{14}$$

The new contribution to the quadrupole moment operator will *not*, however, give rise to enhancement of E2 transitions because J has zero matrix element between distinct states; there is only a small effect arising from terms neglected in (12). Such enhancement would arise from deformations described by the operator $\mathbf{D}\cdot\mathbf{Q}$; unfortunately, however, the properties of this operator are difficult to evaluate.

4. The Effective Hamiltonian

The state ψ_0 and the parameter α will be determined by minimising the expectation value of a Hamiltonian appropriate to shell-model calculations among states of the class (8); ψ_0 is therefore determined as the lowest configuration shell-model wave-function associated with the transformed Hamiltonian,

$$\begin{aligned} H' &= \exp(i\alpha K)H\exp(-i\alpha K) \\ &= H+i\alpha[K,H]+\tfrac{1}{2}(i\alpha)^2[K,[K,H]]+\ldots. \end{aligned} \tag{15}$$

Terms of order α^2 must be retained to find the restoring potential which determines the amount of distortion; we shall assume higher terms may be neglected.

We consider first the contribution of the kinetic energy

$$T = (1/2M)\sum \mathbf{p}^2 \tag{16}$$

to the expression (15). Since J commutes with T,

$$\begin{aligned} i[K,T] &= \tfrac{1}{4}i[\mathbf{D},\mathbf{T}]\cdot\mathbf{J} \\ &= (1/4M)\mathbf{P}\cdot\mathbf{J}. \end{aligned} \tag{17}$$

The matrix elements of this operator can easily be expressed in terms of those of P(or X) for a particular state, conveniently taken as that with $M = J$. We obtain

$$\langle i[K,T]\rangle = (1/2Mb^2)(J+1)(J+\tfrac{3}{2})\langle S\rangle \tag{18}$$

where

$$\langle S\rangle = b^2\langle M = J|P_0|M = J\rangle. \tag{19}$$

Some numerical values are given in the following section.

The second commutator in (15) can easily be evaluated explicitly for lowest configuration oscillator wave-functions. In the commutator

$$i[\mathbf{J}\cdot\mathbf{P},\mathbf{D}\cdot\mathbf{J}] \tag{20}$$

the operator D excites particles into higher shells, and that part of bP which does the same is equal to iD acting on a state of the lowest configuration. In this way we can replace (20) by the equivalent

$$2\mathbf{J}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{J} \tag{21}$$

whose matrix element can be evaluated by a method due to Lipkin [13]), in which use is made of the equivalence

$$2D_qD_{q'} = [D_q,D_{q'}]+\tfrac{1}{4}[D_q,[H_0,D_{q'}]] \tag{22}$$

which holds for matrix elements within the lowest configuration.

The expressions on the right side of (22) are all single-particle operators and the calculation can be completed by standard methods. The final result is that the transformed kinetic energy operator T' has the same matrix

elements within the lowest configuration as the operator

$$T-[\alpha-\tfrac{1}{2}\alpha^2(J-\tfrac{3}{2})(J+\tfrac{5}{2})](J+1)(J+\tfrac{3}{2})(1/2Mb^2)S \\ +(1/2Mb^2)\alpha^2 J(J+1)(J-\tfrac{1}{2})(J+\tfrac{3}{2})(H_0-\tfrac{9}{4}). \tag{23}$$

We have now to consider the contributions to (15) from the *potential* energy. We shall discuss only the mean potential in the form given by Skyrme [1]). The two- and three-body *contact* (velocity-independent) interactions are unaltered by distortion, so that the terms to be considered are

$$W=\sum_{i<j}\sum[\tfrac{1}{4}t_1\delta(\mathbf{r}_i-\mathbf{r}_j)(\mathbf{p}_i-\mathbf{p}_j)^2+\tfrac{1}{4}t_2(\mathbf{p}_i-\mathbf{p}_j)\delta(\mathbf{r}_i-\mathbf{r}_j)\cdot(\mathbf{p}_i-\mathbf{p}_j)]. \tag{24}$$

For the interaction of one particle with a closed shell the matrix elements of $[K, W]$ are proportional to S, like those of $[K, T]$ so that the effect of these terms in $[K, H]$ can be described by an effective mass, for which values are given in the next section.

The most important part of the contribution of the potential energy to the restoring force may be estimated by picking out the terms in the double commutator of H with D_q and D_{-q} which have non-zero average. The result is that W gives (approximately) a contribution to the double commutator proportional to its contribution to the total energy of the system just as T gives a contribution proportional to H_0 in (23).

The approximate form for the transformed Hamiltonian that we shall use well be written then

$$H'=H-(1/2Mb^2)[\alpha(J+1)(J+\tfrac{3}{2})xS-\alpha^2J(J+1)(J-\tfrac{1}{2})(J+\tfrac{3}{2})R] \tag{25}$$

where x is a constant incorporating the contribution of the potential energy W and

$$\langle R\rangle = 2Mb^2\langle T+W\rangle. \tag{26}$$

5. Values of Matrix Elements

(a) For two particles in the p-shell, the operator S has the following non-zero matrix elements

$J=1$	$(^3D, ^3D)$	$-7/10$
	$(^3S, ^3D)$	$-2/5\sqrt{3}$
	$(^1P, ^1P)$	$+1$
	$(^3P, ^3P)$	$-\frac{1}{2}$
$J=2$	$(^3D, ^3D)$	-1
	$(^3P, ^3P)$	$+1$
	$(^1D, ^1D)$	-2
$J=3$	$(^3D, ^3D)$	-2

For two *holes* in the p-shell the sign of S must be changed.

(b) The constant x is a measure of 'reciprocal effective mass'. With the notation and values of constants adopted in ref.[1]) we find:

one particle outside s^4: $x = 1+4.255y^3\lambda_2 \approx 0.84$,

one hole in s^4p^{12}: $x = 1+4.255y^3(\lambda_1+1.5\lambda_2) \approx 1.73$.

In large nuclei x will approach the value $M/M^* = 1.62$.

(c) The following values of the quantity R, defined by eq. (26), have been taken from our analysis of binding energies [1]):

Mass Number	5	6	14	15
$\frac{1}{2}H_0$	8.5	11	31	33.5
R	11.2	19.3	47.3	51.7

The ratio $R/\frac{1}{2}H_0$ tends also towards the limiting value of 1.62.

6. Application to Masses 6 and 14

We consider first the case of mass 14. The ground state of the s^4p^{10} configuration is a mixture of ^{3}S, ^{3}D and ^{1}P states with $J = 1$ and $T = 0$, predominantly ^{3}D. C^{14} has a $J = 0$, $T = 1$ ground state which will be unaffected by the type of distortion considered here; also the operator for an allowed beta-decay is unaltered to first order by the distortion transformation, so that the arguments first given by Inglis [7]) concerning the ratio of S and P states in N^{14} necessary for the vanishing of the beta-decay matrix element are still valid.

Accordingly the wave function ψ_0 characterising the ground state of N^{14} should have the general character determined by Elliott [4]) and Visscher and Ferrell [5]), and we have used the values given by the latter authors (eq. (23) of ref.[5])),

$$S : D : P = 0.173 : 0.920 : 0.355 \tag{27}$$

to determine the magnitude of the distortion. According to eq. (25) the energy will be a minimum where α has the value such that

$$\alpha J(J-\tfrac{1}{2}) = \langle xS\rangle/2\langle R\rangle. \tag{28}$$

For the wave-function (27) the values listed in the previous section give

$$\alpha = 0.027 \ (N^{14} \text{ ground state}). \tag{29}$$

This value does not vary by more than 10 % over a reasonable range of ground state wave-functions. It corresponds, according to eq. (14), with a prolate distortion having $\beta = 0.09$.

The off-diagonal ^{3}S, ^{3}D matrix element in H' then has the value

$$-(1/2Mb^2)\alpha \cdot 5x(\tfrac{2}{5}\sqrt{\tfrac{1}{4}}) = -1.6 \text{ MeV}. \tag{30}$$

This is a little smaller than the value actually used by Visscher and Ferrell. In a similar analysis that we have made with slightly different parameters, the matrix element required was a little smaller and the value of α slightly greater so that the description was more nearly self-consistent.

The energy shift for the N^{14} ground state due to distortion is equal to -0.8 MeV approximately.

The situation in mass 6 is quite different. The ground state of Li^6 is predominantly ^3S, as evidenced by the small quadrupole moment, which implies a D-wave amplitude less than 0.05. For the ^3S state the type of distortion we have considered will not be effective, and therefore no tensor-like interaction will be generated. But if there is a small (genuine) tensor interaction coupling S and D states distortion will tend to amplify it, so that the effective tensor matrix element will be almost double that due to the tensor-force proper; this indicates that the tensor force cannot contribute more than about -0.1 MeV to the ^3S, ^3D matrix element.

On the other hand the states in Li^6 that are mostly D will be affected by appreciable distortions; these would appear to produce level shifts of the order of 1 MeV.

7. Quadrupole Moments in the p-Shell

The quadrupole moment of a self-conjugate $T = 0$ nucleus may be discussed in an especially simple way for the transformation (6). The electric quadrupole operator E_0, eq. (9), may then be replaced by $\frac{1}{2}X_0$, eq. (2), and within the lowest configuration this has the same matrix elements as $\frac{1}{2}b^2S$, eq. (19).

The distortion gives an additional quadrupole moment proportional to α and this in turn, eq. (28) is also proportional to the matrix element of S. Thus for a self-conjugate nucleus, $E'_0 = E_0 + \Delta E_0$ is equivalent to the operator

$$\tfrac{1}{2}b^2S + (xS/2R)\sum \mathbf{r}^2 = \tfrac{1}{2}b^2S(1 + (\tfrac{1}{2}H_0/R)x) \qquad (31)$$

since the expectation value of $\sum \mathbf{r}^2$ is equal to $\frac{1}{2}H_0$. According to section 5(c), the factor multiplying E_0 is nearly equal to 2.

Unfortunately there is no clear evidence concerning the three nuclei of interest, Li^6, B^{10}, N^{14}. The moment of Li^6 is very small anyway. The moment of B^{10} according to the shell model [9]) is very sensitive to the details of the interaction assumed. The shell-model prediction for the moment of N^{14} is however fairly definite, at about 11 mb. The distortion considered in this paper would increase it to about 20 mb; on the other hand the latest theoretical calculation of atomic wave-functions by Bassompierre [14]) indicates the small value of 7 mb.

8. Conclusions

We have suggested that the tensor-force explanation of the long-lifetime of C^{14} is unacceptable, and that the explanation should therefore be sought in configuration admixture; this conclusion is supported by recent evidence from stripping reactions [15]), which however points to an explanation different from that given here. We have considered one particular type of deformation operator in conjunction with a reasonable shell-model Hamiltonian, and shown that applied to the N^{14} ground state this provides an apparent tensor interaction with sign and magnitude agreeing with those postulated in the shell-model analyses of the system.

However, the type of distortion we have considered is rather special, and we can only claim at present that distortion effects are qualitatively capable of providing an explanation. The next stage in the investigation must involve operators more closely related to those considered by Elliott [11]), which are capable of describing intrinsic distortions in spin zero states and which might account for the enhancement of E2 transitions. While such operators have very simple properties in relation to the oscillator Hamiltonian, their relations to a more realistic one (including spin-orbit coupling) are at present obscure.

With regard to the potential effective between nucleons in nuclear matter there does not appear to be any unambiguous evidence supporting the presence of tensor interactions, and it is clear that they cannot be determined empirically until the shell-model wave-functions are better understood.

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QUANTAL DERIVATION OF MASS DEFECT

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The mass defect of the nucleus is generally understood, according to Einstein's principle of the mass-energy equivalence, as the decrease of the total energy which makes the constituent nucleons into a bound system, the nucleus. This interpretation is of fundamental importance in nuclear physics. In the present note it will be shown that this phenomenon is really expressible in the explicit form of the Pauli type equation. The mass-energy equivalence itself is a relativistic notion, although the idea of the mass defect is often applied to the non-relativistic phenomena. Therefore the discussion should be at least semi-relativistic, and the following consideration will be based on this viewpoint.

We consider a system of two Dirac particles of rest mass m_1 and m_2. In the conventional way the eigenvalue equation of the system is given by

$$(H-E)\psi = 0 \tag{1}$$

where H denotes the Hamiltonian without the rest energy of two particles:

$$H = -ic\rho_1^{(1)}\mathrm{d}_1 - ic\rho_1^{(2)}\mathrm{d}_2 + m_1c^2(\rho_3^{(1)}-1) + m_2c^2(\rho_3^{(2)}-1) + H_{\mathrm{int}}, \tag{2}$$

$$\mathrm{d}_1 = \hbar\sigma^{(1)}\cdot\nabla_1, \qquad \mathrm{d}_2 = \hbar\sigma^{(2)}\cdot\nabla_2. \tag{3}$$

The superscripts of Dirac's matrices and the subscripts of nablas indicate the particles. The ρ-matrix dependence of the interaction H_{int} between the particles is assumed as

$$H_{\mathrm{int}} = U + \rho_1^{(1)}W_1 + \rho_1^{(2)}W_2 + \rho_1^{(1)}\rho_1^{(2)}V \tag{4}$$

where U, W_1, W_2, and V may depend on σ-matrices. This is the general form for the present consideration. We denote the eigenmatrices of ρ_3 by χ^+ and χ^- and expand ψ as follows:

$$\psi = \psi^{++}\chi^+(1)\chi^+(2) + \psi^{+-}\chi^+(1)\chi^-(2) + \psi^{-+}\chi^-(1)\chi^+(2) + \psi^{--}\chi^-(1)\chi^-(2) \tag{5}$$

where the superscripts of χ's represent the signs of eigenvalues of ρ_3 and their arguments indicate the particles. The equation (1) represents simultaneous equations for ψ^{++} etc. If we eliminate the smaller components from them we have the following equation for the largest component ψ^{++}:

$$(H_0+H')\psi^{++} = E\psi^{++}. \tag{6}$$

This is the Pauli type equation, in which all quantities are formally expanded in negative powers in c, and terms up to c^{-2} are only retained. Its approximate Hamiltonian is given by

$$H_0 = -\left(1-\frac{E}{2m_1c^2}\right)\frac{\hbar^2}{2m_1}\Delta_1-\left(1-\frac{E}{2m_2c^2}\right)\frac{\hbar^2}{2m_2}\Delta_2+U, \tag{7}$$

$$\begin{aligned} H' = &-\frac{\mathrm{d}_1 U \mathrm{d}_1}{4m_1{}^2c^2}-\frac{\mathrm{d}_2 U \mathrm{d}_2}{4m_2{}^2c^2}+\frac{1}{2m_0c^2}\left(\frac{\mathrm{d}_1\mathrm{d}_2}{2m}-V\right)^2-\frac{\mathrm{d}_1 V \mathrm{d}_2+\mathrm{d}_2 V \mathrm{d}_1}{4mm_0c^2} \\ &-\frac{i}{2m_1c}(\mathrm{d}_1 W_1+W_1\mathrm{d}_1)-\frac{i}{2m_2c}(\mathrm{d}_2 W_2+W_2\mathrm{d}_2)+\frac{W_1{}^2}{2m_1c^2}+\frac{W_2{}^2}{2m_2c^2} \end{aligned} \tag{8}$$

where $m_0 = m_1+m_2$, and $m^{-1} = m_1{}^{-1}+m_2{}^{-1}$. If H_{int} has additional terms involving ρ_2 and ρ_3 the general form of H' is still unchanged: the only difference is that new terms appear with different factors -1 or $\pm i$.

We introduce the centre-of-mass and relative coordinates $x_0 = (m_1x_1+m_2x_2)/m_0$ and $x = x_1-x_2$ etc., and we denote their Laplacians by Δ_0 and Δ respectively. For the non-relativistic motion of the centre of mass the eigenvalue equation (6) reduces to

$$\left(-\frac{\hbar^2}{2M}\Delta_0-\frac{\hbar^2}{2\mu}\Delta+U+H''\right)\psi^{++} = E\psi^{++} \tag{9}$$

where

$$M = m_1+m_2+(E/c^2), \qquad \mu = m\left\{1+\frac{E}{2m_0c^2}\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\right\} \tag{10}$$

and H'' is obtained from H' by replacing ∇_1 and ∇_2 by ∇ and $-\nabla$ respectively. The quantal equations (9) and (10) express the fact that the decrease of the rest mass of the total system is caused by an internal energy of negative value. We can interpret H'' as the relativistic correction for the internal Hamiltonian of the system, while M and μ represent the total and reduced mass of the system respectively. If E_i and K are the internal energy of the system and the kinetic energy of the centre-of-mass motion respectively, E is equal to E_i+K. The mass defect is given by

$$\Delta M = m_1+m_2-\lim_{K\to 0} M = -E_i/c^2.$$

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A NOTE ON THE DECAY OF THE Σ-HYPERON AND ITS ANTIPARTICLE

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Abstract: This note examines the information which may be derived from measurements of the branching ratios and angular correlation of Σ- and $\bar{\Sigma}$-decays.

1. Branching Ratio

Recently Okubo in a very interesting paper [1]) proposed an experiment on the measurement of the branching ratio in the decay of the Σ^+ hyperon and its antiparticle $\bar{\Sigma}^+$. A difference between their branching ratios will indicate the violation of invariance both under charge conjugation and under time reversal.

To obtain his results Okubo utilized the lowest order perturbation theory for the weak interaction. In view of the great importance of this experiment as a test of the validity of invariance under time reversal we shall show in this note that Okubo's results are quite general. They can be derived from the Lüders-Pauli theorem together with the requirement of unitarity of the S-matrix.

The probability of the decay process is characterized by the following elements of the R-matrix

$$\langle i, j, l, m|R|j, m\rangle$$

where i denotes the total isobaric spin of the nucleon-meson state; j the total angular momentum of the system which is equal to the spin of the Σ-particle; m the eigenvalue of the operator j_z; l the orbital angular momentum of the nucleon-meson state; and R is connected with the scattering matrix S by the relation $S = 1-iR$.

Okubo's derivation rests on the relation (7) in his paper [1]) which can be expressed in the form

$$\langle i, j, l, m|R|j, m\rangle_c = I_0(-1)^l \exp(2i\delta_{i,j,l})\langle i, j, l, m|R|j, m\rangle^* \qquad (1)$$

where the subscript c on the left-hand side denotes the matrix elements for the antiparticle process; I_0 the product of the inner parities of the Σ, N particles and the π-meson; $\delta_{i,j,l}$ the phase shift of the π—N scattering in the

corresponding state. The relation (1) is proved in ref.[1]) by considering only the lowest order perturbation in the weak interaction.

Applying the ICT-theorem to the transition matrix elements we obtain according to ref.[2])

$$\langle i, j, l, m|R|j, m\rangle_c = -I_0(-1)^l\langle i, j, l, -m|R^+|j, -m\rangle^* = -I_0(-1)^l\langle i, j, l, m|R^+|j, m\rangle^* \quad (2)$$

where R^+ denotes the hermitian conjugate operator of R. The unitarity condition for the S-matrix leads to the relation

$$-i(R^+-R) = R^+R. \quad (3)$$

Multiplying both sides of (3) on the left by the state $\langle i, j, l, m|$ and on the right by $|j, m\rangle$ we get

$$-i\langle i, j, l, m|R^+-R|j, m\rangle = [\exp(-2i\delta_{i,j,l})-1]\langle i, j, l, m|R|j, m\rangle. \quad (4)$$

In obtaining (4) we note that the only possible intermediate state which gives a contribution to the matrix element $\langle i, j, l, m|R^+R|j, m\rangle$ is the nucleon-meson state if the electromagnetic and other weak interaction effects are neglected. This neglect does not correspond to the lowest order perturbation theory, since we are now dealing with the R-matrix, whose elements are connected directly with the observed probability of the corresponding process. A simple calculation shows that the contribution of the neglected intermediate states (e.g. nucleon+meson+photon) is at least of the order $\alpha = 1/137$ times smaller than the main contribution.

Similarly for the antiparticle process we have

$$-i\langle i, j, l, m|R^+-R|j, m\rangle_c = [\exp(-2i\delta_{i,j,l})-1]\langle i, j, l, m|R|j, m\rangle_c. \quad (5)$$

The phase shifts for the $\pi-\bar{N}$ scattering are taken equal to that of $\pi-N$ scattering. Applying (2) to the left-hand side of (4) and (5) we get

$$\langle i, j, l, m|R^+-R|j, m\rangle_c = I_0(-1)^l\langle i, j, l, m|R^+-R|j, m\rangle^* \quad (6)$$

and the relation (1) follows directly from (4), (5) and (6).

2. Angular Correlation

Measurement of the angular correlation in the decay of the Σ-particle and its antiparticle will also yield valuable information on the validity of invariance under charge conjugation and under time reversal. The simplest case will be the decay of Σ^- and its antiparticle. The asymmetry parameter α defined in ref.[3]) has the form

$$\alpha = \mathrm{Re}\, A^*B/(|A|^2+|B|^2)$$

with

$$A = \langle \tfrac{3}{2}, \tfrac{1}{2}, 0, m|R|\tfrac{1}{2}, m\rangle$$
$$B = \langle \tfrac{3}{2}, \tfrac{1}{2}, 1, m|R|\tfrac{1}{2}, m\rangle \quad (7)$$

where, for simplicity, the spin of the Σ-particle is assumed to be $\frac{1}{2}$. The asymmetry parameter for the antiparticle is denoted by α_c. From (1) and (7) we get

$$\frac{\alpha_c}{\alpha} = -\frac{\cos(\delta_s-\delta_p-\Delta_s+\Delta_p)}{\cos(\delta_s-\delta_p+\Delta_s-\Delta_p)} \tag{8}$$

where we have denoted the phase shifts of the s-state and the p-state by δ_s and δ_p respectively; Δ_s and Δ_p are defined by the relation

$$A = |A| \exp[i(\delta_s+\Delta_s)], \quad B = |B| \exp[i(\delta_p+\Delta_p)].$$

The π—N phase shifts are given by ref.[4]):

$$\delta_s-\delta_p = -17^\circ \qquad (\text{at } E \approx 120 \text{ MeV}).$$

If the degree of polarization of Σ^- and its antiparticle can be determined independently and the deviation from I, T and C invariance is quite large, the deviation of α_c/α from unity may be detected by experiment. A deviation of α_c/α from unity will indicate violation of both T and C invariance.

Formula (8) retains its form when the spin of the Σ particle is $\frac{3}{2}$. In this case δ_s, δ_p, Δ_s and Δ_p should be replaced by the corresponding quantities in the states $i=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$ and $i=\frac{3}{2}$, $j=\frac{3}{2}$, $l=2$, respectively. A similar but more complicated formula holds for the decay of Σ^+ and its antiparticle.

Finally, we note that the lack of asymmetry in the decay of the Σ^--particle [5]) may be due to the accidental equality $\delta_s-\delta_p+\Delta_s-\Delta_p \approx \pm 90^\circ$. In this case we may expect quite large asymmetry in the decay of its antiparticle.

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ON THE SUPERFLUID STATE OF THE ATOMIC NUCLEUS

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Abstract: The variation principle proposed by N. N. Bogolubov and the mathematical methods developed in the theory of superconductivity are employed to study the properties of the atomic nucleus. Weak interactions between protons (or neutrons) located in the same shell are treated on the basis of a nuclear model in which nucleons of the closed internal shells are assumed to produce a centrally symmetrical field which is somewhat distorted by nucleons in the outer shell. It is shown that the interaction between protons in an outer shell give rise to a superfluid state of the nucleus, the main role being played by interactions between protons with equal and opposite projections of the angular momenta on the nuclear symmetry axis. The energy of the ground superfluid state and first excited state is evaluated and it is shown that in the case of an even number of protons in the shell, an energy gap between the superfluid and first excited states should exist, whereas no such gap exists if the number of protons is odd. This permits one to explain the energy gap in heavy even nuclei. Conditions for the appearance of a superfluid state in an atomic nucleus have been obtained. They reduce to the requirement that attractive forces predominate between interacting protons located in the same shell near the Fermi surface energy. The critical temperature for the phase transition of a nucleus from the superfluid to normal state has been determined.

1. Introduction

As is well known [1]), superconductivity of metals is the result of weak interactions between electrons possessing equal and opposite momenta near the Fermi surface. The superconductive state is energetically lower than that of a purely degenerate Fermi-gas which will be called the normal state. Proceeding from the fact that Fermi-systems in nuclei and metals are similar in certain respects [2]), we shall consider nucleon interactions in complex nuclei and investigate the possibility of existence of such states of the atomic nucleus which are energetically lower than the normal state. We shall call this state the superfluid state of the atomic nucleus.

The appearance of a superfluid state in the atomic nucleus will be investigated with the help of the Bogolubov variational principle [3]) (which is a generalization of the well-known Fock method [4])) and also by means of mathematical methods developed in the theory of superconductivity [1]).

The nuclear shell model will be used to investigate weak interactions of protons (or neutrons) located in the same shell near the Fermi surface in heavy and medium nuclei. The proton state will be characterized by a quantum number s, which specifies the shell and a quantum number m,

which specifies the projection of the angular momentum on the symmetry axis of the nucleus. The nuclear field is assumed to deviate somewhat from central symmetry, and therefore there will be no energy degeneracy with respect to m.

2. Ground Superfluid State

Let us consider weak interactions between protons (or neutrons) which are situated on the outer shell of the nucleus. Nucleons forming the closed internal shells create a centrally-symmetric field which is somewhat distorted by nucleons of the outer shell.

The Hamiltonian describing the interaction of protons located in a given shell may be written as follows:

$$H = \sum_{s,m,\sigma} \{E(s,m) - E_F\} a^+_{m\sigma}(s) a_{m\sigma}(s) + \frac{1}{2N} \sum_{\substack{(s,\sigma_1,\sigma_2,m_1,m_2,m'_1,m'_2 \\ m_1+m_2=m'_1+m'_2;\ m_1\neq m'_1)}} J(s|\sigma_1 m_1, \sigma_2 m_2; \sigma_1 m'_1, \sigma_2 m'_2) a^+_{m_2\sigma_2}(s) a^+_{m_2\sigma_1}(s) \cdot a_{m'_2\sigma_2}(s) a_{m'_1\sigma_1}(s), \tag{1}$$

where N is the number of levels, $\sigma = \pm 1$ characterizes the sign of m, $a^+_{m\sigma}$, $a_{m\sigma}$ are proton production and absorption operators. $E(s,m) = E(s,-m)$ is the proton energy in the state s, m. E_F is a parameter which plays the role of chemical potential; in the normal state it is equal to the Fermi surface energy. The function $J(s|m_1, m_2; m_1, m'_2)$ is real and possesses the following properties:

$$J(s|m_1, m_2; m'_1, m'_2) = J(s|m'_1, m'_2; m_1, m_2),$$
$$J(s|m_1, m_2; m'_1, m'_2) = J(s|-m_1, -m_2, -m'_1, -m'_2). \tag{2}$$

Let us perform a linear canonical transformation of the Fermi amplitudes

$$a_{m,\sigma}(s) = u_m(s)\alpha_{m,-\sigma}(s) + \sigma v_m(s)\alpha^+_{m\sigma}(s). \tag{3}$$

In order not to violate their commutation properties the condition

$$\eta_m(s) = u_m(s)^2 + v_m(s)^2 - 1 = 0 \tag{4}$$

must be fulfilled. We define a new vacuum state C_0:

$$\alpha_{m\sigma} C_0 = 0, \tag{5}$$

and find the mean value of H over the new vacuum state:

$$\bar{H} = \langle C_0^* H C_0 \rangle = 2 \sum_{s,m} \{E(s,m) - E_F\} v_m(s)^2 + \frac{1}{N} \sum_{\substack{s,m,m' \\ m\neq m'}} \{J(s|m, -m; m', -m') u_m(s) v_m(s) u_{m'}(s) v_{m'}(s) - J(s|m, m'; m', m) v_m(s)^2 v_{m'}(s)^2\} \equiv \mathcal{E}(u, v). \tag{6}$$

The functions $u_m(s)$, $v_m(s)$ can be determined from the condition of minimum of the form $\mathscr{E}(u, v)$ with the auxiliary condition (4). The corresponding equation is

$$\delta\tilde{\mathscr{E}}(u, v) = \delta\{\mathscr{E}(u, v)+\sum_{s, m}\lambda_m(s)\eta_m(s)\} = 0, \tag{7}$$

where $\lambda_m(s)$ is the Euler factor, the variations $\delta u_m(s)$ and $\delta v_m(s)$ being considered as independent. As a result we obtain

$$\xi_m(s)u_m(s)v_m(s) + \frac{u_m(s)^2-v_m(s)^2}{2N}\sum_{m'}J(s|m, -m; m', -m')u_{m'}(s)v_{m'}(s) = 0, \tag{8}$$

where

$$\xi_m(s) = E(s, m)-E_F-\frac{1}{N}\sum_{m'}J(s|m, m'; m', m)v_{m'}(s)^2. \tag{9}$$

$\mathscr{E}(u, v)$ gives the energy of the ground state for $u_m(s)$, $v_m(s)$ satisfying (8). Eq. (8) admits the solution

$$u_m(s) = 1-\theta_F(s, m), \quad v_m(s) = \theta_F(s, m), \tag{1}$$

which corresponds to the normal state. The function $\theta_F(s, m) = 1$ if $E(s, m) < E_F$ and $\theta_F(s, m) = 0$ if $E(s, m) > E_F$.

To solve (8) we introduce a new function

$$c_m(s) = \frac{1}{N}\sum_{m'}J(s|m, -m; m', -m')u_{m'}(s)v_{m'}(s)$$

related to $u_m(s)$ and $v_m(s)$ in the following way:

$$u_m(s)^2 = \tfrac{1}{2}\left[1+\frac{\xi_m(s)}{\tilde{\varepsilon}_m(s)}\right], \quad v_m(s)^2 = \tfrac{1}{2}\left[1-\frac{\xi_m(s)}{\tilde{\varepsilon}_m(s)}\right], \tag{11}$$

$$u_m(s)v_m(s) = \tfrac{1}{2}\sqrt{1-\frac{\xi_m(s)^2}{\tilde{\varepsilon}_m(s)^2}}, \quad u_m(s)v_m(s) = -\tfrac{1}{2}\frac{c_m(s)}{\tilde{\varepsilon}_m(s)}, \tag{12}$$

$$\tilde{\varepsilon}_m(s) = \sqrt{c_m(s)^2+\xi_m(s)^2}.$$

For $c_m(s)$ we obtain the equation

$$c_m(s) = -\frac{1}{2N}\sum_{m'}J(s|m, -m; m', -m')\frac{c_m(s)}{\sqrt{c_{m'}(s)^2+\xi_{m'}(s)^2}}. \tag{13}$$

Consider now the interaction of protons located in an outer shell characterized by $s = s_0$ for which the proton energy $E(s_0, m)$ is in the neighbourhood of the Fermi surface, i.e.,

$$E_F-\Delta_1 \leqq E(s_0, m) \leqq E_F+\Delta_2$$
$$\Delta_1 \ll E_F, \quad \Delta_2 \ll E_F.$$

Changing in (13) from the sum to the integral we obtain

$$c_m(s_0) = -\tfrac{1}{2}\int_{m_1}^{m_2} \mathrm{d}m' \rho_0(m') J(s_0|m, -m; m', -m') \frac{c_m(s_0)}{\sqrt{c_{m'}(s_0)^2+\xi_{m'}(s_0)^2}}, \quad (13')$$

where $\rho_0(m') = \mathrm{d}n/\mathrm{d}m'$ is the level density,

$$E(s_0, m_1) = E_F - \Delta_1, \quad E(s_0, m_2) = E_F + \Delta_2, \quad E(s_0, m_0) = E_F.$$

To obtain the asymptotic form of the solution of (13') for small values of J we consider the approximate equation

$$c_m(s_0) = c_{m_0}(s_0) \ln \frac{c_{m_0}(s_0)}{\mu} \frac{\rho_0(m_0)}{\left\{\dfrac{\mathrm{d}\xi_{m'}(s_0)}{\mathrm{d}m'}\right\}_{m'=m_0}} J(s_0|m, -m; m_0, -m_0) \quad (14)$$

$$+ \tfrac{1}{2}\int_{m_1}^{m_2} \mathrm{d}m' \ln \frac{|E(s_0, m') - E_F|}{\mu} \frac{\mathrm{d}}{\mathrm{d}m'} \left[J(s_0|m, -m; m', -m') c_{m'}(s_0) \frac{\rho_0(m')}{\dfrac{\mathrm{d}\xi_{m'}(s_0)}{\mathrm{d}m'}} \right],$$

which at small J coincides with (13') asymptotically. Here μ is the proton mass. The approximate solution of (14) at small $c_m(s_0)$ may be found in the following form [6])

$$c_m(s_0) = \omega \frac{J(s_0|m, -m; m_0, -m_0)}{J(s_0|m_0, -m_0; m_0, -m_0)} e^{\frac{1}{G}}, \quad (15)$$

where

$$G = J(s_0|m_0, -m_0; m_0, -m_0) \frac{\rho_0(m_0)}{\left\{\dfrac{\mathrm{d}\xi_{m'}(s_0)}{\mathrm{d}m'}\right\}_{m'=m_0}}, \quad (16)$$

$$\ln \frac{\omega}{\mu} = \int_{m_1}^{m_2} \mathrm{d}m' \frac{\left\{\dfrac{\mathrm{d}\xi_m(s_0)}{\mathrm{d}m}\right\}_{m=m_0}}{\rho_0(m_0)} \ln \frac{|E(s_0, m') - E_F|}{\mu} \cdot \frac{\mathrm{d}}{\mathrm{d}m'} \left[J(s_0|m_0 - m_0; m', -m') c_{m'}(s_0) \frac{\rho_0(m')}{\dfrac{\mathrm{d}\xi_{m'}(s_0)}{\mathrm{d}m'}} \right]. \quad (17$$

It can be seen from the foregoing calculations that to obtain asymptotic quantities pertaining to the superfluid state of an atomic nucleus only a small part of the total interaction $J(s|m_1, m_2; m'_1, m'_2)$ between protons located in a given shell is essential, namely, $J(s_0|m, -m; m', -m')$, i.e., the interaction between protons with equal and opposite projections of the angular momentum on the symmetry axis of the nucleus. The remaining part of the proton interactions should be taken into account by perturbation theory methods. In further investigations we shall therefore use the interaction

$J(s|m, -m; m', -m')$ which will be denoted as $J(s|m, -m; m', -m') \equiv J(s|m, m')$.

3. Ground State and Excited States

Let us calculate the energy of the excited state by taking into account the weak interactions between protons (or neutrons) located on the same shell and possessing equal and opposite values of the angular momentum projection on the symmetry axis of the nucleus.

In the approximation under consideration $\xi_m(s)$ changes into $E(s, m)-E_F$, and $\bar{\varepsilon}_m(s)$ into $\varepsilon_m(s) = \sqrt{\{E(s, m)-E_F\}^2+C_m(s)^2}$. The expression $\rho_0(m')/\mathrm{d}\xi_{m'}(s)/\mathrm{d}m'$ becomes equal to $\rho(E) = \mathrm{d}n/\mathrm{d}E$, which for $s = s_0$, $m = m_0$ is the level density near the Fermi surface. Further let us write the expression for $c_m(s_0)$ in the form

$$c_m(s_0) = \omega \frac{J(s_0|m, m_0)}{J(s_0|m_0, m_0)} \mathrm{e}^{\frac{1}{G}}, \tag{15'}$$

where

$$G = J(s_0|m_0, m_0)\rho(E), \tag{16'}$$

$$\ln \frac{\omega}{\mu} = \int_{m_1}^{m_2} \frac{\mathrm{d}m'}{\rho(E_F)} \ln \frac{|E(s_0, m')-E_F|}{\mu} \frac{\mathrm{d}}{\mathrm{d}m'} [J(s_0|m_0, m')c_{m'}(s_0)\rho(E')]. \tag{17'}$$

Calculating the mean energy in the excited state $C_1 = \alpha_{m0}^+ C_s = \alpha_{m1}^+ C_s$ (in the following $\alpha_{m0} \equiv \alpha_{m-}$; $\alpha_{m1} = \alpha_{m+}$) we obtain

$$\begin{aligned}
&\langle C_s^* \alpha_{m1}(s) H \alpha_{m1}^+(s) C_s\rangle \\
&\quad = 2\sum_{m'}\{E(s, m')-E_F\}v_{m'}(s)^2+\{E(s, m)-E_F\}\{u_m(s)^2-v_m(s)^2\} \\
&\quad + \frac{1}{N}\sum_{m'm''} J(s|m', m'')u_{m'}(s)v_{m'}(s)u_{m''}(s)v_{m'''}(s) \\
&\quad - \frac{2}{N}\sum_{m'} J(s|m, m')u_{m'}(s)v_{m'}(s)u_m(s)v_m(s).
\end{aligned} \tag{18}$$

The difference $\Delta E'$ between the energies in states C_1 and C_s is found in the form

$$\Delta E'_m(s) = \langle C_s^* \alpha_{m1}(s) H \alpha_{m1}^+(s) C_s\rangle - \langle C_s^* H C_s\rangle = \varepsilon_m(s). \tag{19}$$

It is worth noting that $C_1 = \alpha_{m0}^+ C_s$ is a superposition of states with odd numbers of particles. Therefore, to avoid the transition from an even nucleus to an odd-mass one the calculations should be performed for the excited state $C_2 = \alpha_{m1}^+ \alpha_{m0}^+ C_s$.

Calculating the difference ΔE^{I} between the energy of the first excited and ground superfluid state of the even nucleus we find [7])

$$\Delta E_m^{\mathrm{I}}(s) = \langle C_s^* \alpha_{m0}(s)\alpha_{m1}(s) H \alpha_{m1}^+(s)\alpha_{m1}^+(s) C_s\rangle - \langle C_s^* H C_s\rangle = 2\varepsilon_m(s) \tag{20}$$

and for $s = s_0$, $m = m_0$

$$\Delta E^1_{m_0}(s_0) \approx 2\omega e^{\frac{1}{G}}. \tag{20'}$$

It can be seen from here that the first excited state is separated from the ground state by the gap (20'). It should be noted that the energy gap should not appear in odd-mass nuclei since transitions between the states inside the outer shell will occur; there should also be no energy gap in the excitation of the normal state in even-mass as well as in odd-mass nuclei.

The difference ΔE between the superfluid and the normal states may be obtained as follows [7]):

$$\Delta E_m(s) = \langle C_s^* H C_s\rangle - \langle C_n^* H C_n\rangle$$
$$= -\tfrac{1}{2}\varepsilon_m(s) - \tfrac{1}{2}\frac{\{E(s,m)-E_F\}^2}{\varepsilon_m(s)} + \{E(s,m)-E_F\}\{1-2\theta_F(s,m)\}, \tag{21}$$

at $s = s_0$, $m = m_0$

$$\Delta E_{m_0}(s_0) = -\tfrac{1}{2}\varepsilon_{m_0}(s_0). \tag{21'}$$

It can be seen from here that the superfluid state is energetically lower and is separated from the normal state by a gap.

Thus, the interactions between protons located in the same shell and possessing equal and opposite angular momentum projections lead to the appearance of a superfluid state in the atomic nucleus. The presence of an energy gap between the first excited state and the ground superfluid one is confirmed by the arguments of A. Bohr, Mottelson and Pines [2]), who considered the possibility of explaining the energy gap in heavy even nuclei in such a way.

4. Conditions for Occurrence of the Superfluid State in Atomic Nuclei

Using the Bogolubov variational principle one may obtain the conditions for occurrence of the superfluid state in a system. In ref.[8]) a method for investigating the superfluidity condition for nuclear matter was proposed, whereas in ref.[5]) it was shown that nuclear matter possesses the superfluidity property if attractive forces predominate at the Fermi surface energy. Let us investigate the conditions for the appearance of the superfluid state in the nuclear model chosen by us above.

In order to determine under what conditions the energy of the normal state C_n is not minimum, and, therefore, the superfluid state C_s will appear, let us find the second variation of $\mathscr{E}(u, v)$ assuming condition (4) fulfilled, i.e.

$$\delta^2\tilde{\mathscr{E}}(u, v) = \delta^2\{\mathscr{E}(u, v) + \sum_{s,m}\lambda_m(s)\eta_m(s)\}. \tag{22}$$

If expression (22) is less than zero for a solution corresponding to the normal state C_n, the superfluid state C_s should appear. As a result of the calculations we find that the superfluid state of the atomic nucleus exists if such solutions of the equation

$$2|E(s_0, m)-E_F|\varphi_m(s_0)+\frac{1}{N}\sum_{m'}{}'J(s_0|m, m')\varphi_{m'}(s_0) = E\varphi_m(s_0) \qquad (23)$$

exist which have negative eigenvalues $E = -2\delta$, $\delta > 0$.

Let us investigate the asymptotic form of solutions of (23) for J tending to zero and E also tending to zero remaining negative all the time.

Assuming

$$\varphi_m(s) = \frac{\theta_{m'}(s)}{|E(s, m)-E_F|+\delta}$$

and changing from the sum to an integral we obtain

$$2\theta_m(s_0)+\int_{m_1}^{m_2} dm' \rho_0(m') J(s_0|m, m') \frac{\theta_m(s_0)}{|E(s_0, m')-E_F|+\delta} = 0. \qquad (24)$$

Taking into account that for $\delta \to 0$ the integral in (24) is logarithmically divergent near the Fermi surface we pass to consideration of the approximate equation

$$\theta_m(s_0)+\ln\frac{\Delta}{\delta}\rho(E_F)J(s_0|m, m_0)\theta_{m_0}(s_0)$$
$$-\tfrac{1}{2}\int_{m_1}^{m_2} dm' \ln\frac{|E(s, m')-E_F|}{\mu}\frac{d}{dm'}[J(s_0|m, m')\rho(E')\theta_{m'}(s_0)] = 0, \qquad (25)$$

which at small J coincides asymptotically with (24); in (25) $\Delta = \Delta_1 = \Delta_2 \gg \delta$. Let us introduce a new function

$$f_m(s_0) = \frac{\theta_m(s_0)}{\theta_{m_0}(s_0)}\frac{1}{\ln\frac{\Delta}{\delta}},$$

$$f_{m_0}(s_0) = \frac{1}{\ln\frac{\Delta}{\delta}} > 0.$$

The equation for $f_m(s_0)$ with $m = m_0$ may be obtained as follows:

$$f_{m_0}(s_0)+\rho(E_F)J(s_0|m_0, m_0)$$
$$-\tfrac{1}{2}\int_{m_1}^{m_2} dm' \ln\frac{|E(s, m)-E_F|}{\mu}\frac{d}{dm'}[J(s_0|m_0, m')\rho(E')f_{m'}(s_0)] = 0. \qquad (26)$$

If J tends to zero eq. (26) has a solution if

$$\rho(E_F)J(s_0|m_0, m_0) < 0. \qquad (27)$$

Indeed, $f_{m_0}(s_0) > 0$, whereas the last term in (26) is a quantity of higher order of smallness, providing $J(s_0|m_0, m')$ does not change rapidly for m' lying between m_1 and m_2. Since the level density $\rho(E_F) > 0$, the superfluidity condition of the atomic nucleus assumes the form

$$J(s_0|m_0, m_0) < 0, \tag{27'}$$

i.e., attractive forces must predominate between protons located in the same shell.

5. Temperature of Phase Transition

To determine the critical temperature of the phase transition of the nucleus from the superfluid to the normal state we make use of the statistical variational principle [9]). The statistical variational principle which is suitable for determination of the thermodynamic quantities both at zero temperature and at temperatures differing from zero is a generalization of the variational principle suggested by Bogolubov [3]).

As a result of the calculations we find that the superfluid state of the atomic nucleus at a temperature θ differing from zero exists in the case when the equation

$$\begin{aligned} &2|E(s_0, m)-E_F|\operatorname{cth}\frac{|E(s_0, m)-E_F|}{2\theta}\psi_m(s_0) \\ &+\frac{1}{N}\sum_{m'}{}'J(s_0|m, m')\psi_{m'}(s_0) = E\psi_m(s_0) \end{aligned} \tag{28}$$

has solutions with negative eigenvalues $E < 0$. It can easily be seen that at $\theta = 0$ equation (28) passes into (23). If the eigenvalue $E > 0$, then the normal state is the ground state. If the superfluid phase exists for $\theta = 0$, one finds that E will increase with the temperature θ. In the transition from the superfluid phase to the normal one at the critical temperature θ_0 the eigenvalue E must vanish. This yields an equation for the critical temperature:

$$\begin{aligned} &2|E(s_0, m)-E_F|\operatorname{cth}\frac{|E(s_0, m)-E_F|}{2\theta_0}\psi_m(s_0) \\ &+\frac{1}{N}\sum_{m'}{}'J(s_0|m, m')\psi_{m'}(s_0) = 0. \end{aligned} \tag{29}$$

Noting that $|E(s_0, m)-E_F|$ is rather small, and passing from the sum to the integral we obtain the approximate equation

$$\psi_m(s_0)+\frac{1}{4\theta_0}\int_{m_1}^{m_2} dm'\rho_0(m')J(s_0|m, m')\psi_{m'}(s_0) = 0. \tag{30}$$

In order that (30) have a non-vanishing solution the determinant D must

vanish. Taking into account the weakness of the interaction we may obtain from the equation $D = 0$ the following expression for the phase transition temperature:

$$\theta_0 = -\tfrac{1}{4}\int_{m_1}^{m_2} \mathrm{d}m' \rho_0(m') J(s_0|m', m') \tag{31}$$

or

$$\theta_0 \approx -\tfrac{1}{4}\bar{\rho}_0 J. \tag{31'}$$

6. Rough Approximation

In section 2 we obtained the asymptotic form of the solution of the basic equation (13). Let us consider two even less precise solutions of this equation.

We somewhat simplify eq. (13') writing it as follows:

$$c_m(s_0) = -\tfrac{1}{2}\int_{E_F-\Delta}^{E_F+\Delta} \mathrm{d}E' \rho(E') J(s_0|m, m') \frac{c_{m'}(s_0)}{\sqrt{(c_{m'}s_0)^2+(E-E_F)^2}}. \tag{32}$$

$\rho(E')$ and $J(s_0|m, m')$ will be assumed to be weakly dependent upon E'. They can then be written before the integral sign and in this case $c_{m'}(s)$ will depend only upon s. As a result we obtain

$$1 = -\tfrac{1}{2}\bar{\rho}J\int_{E_F-\Delta}^{E_F+\Delta} \frac{\mathrm{d}E'}{\sqrt{c(s_0)^2+(E'-E_F)^2}}. \tag{33}$$

Taking into account that $c(s_0) \ll \Delta$ we find that

$$c(s_0) = 2\Delta\, \mathrm{e}^{\frac{1}{\bar{\rho}J}}. \tag{34}$$

In this case

$$\varepsilon_m(s_0) = \sqrt{\{E(s_0, m)-E_F\}^2+4\Delta^2\, \mathrm{e}^{\frac{2}{\bar{\rho}J}}},$$

for $J \to 0$

$$\varepsilon_m(s_0) \approx \Delta\, \mathrm{e}^{\frac{1}{\bar{\rho}J}}.$$

Thus, in this rough approximation there will also be a gap between the excited and the ground superfluid states, as well as between the superfluid and normal states. The superfluidity condition in this case reduces to the requirement $J < 0$, whereas the temperature of the phase transition is determined by formula (31').

As another rough approximation we can consider the case, when $E(s, m)$ is independent of m (degeneracy with respect to the quantum number m). Then for a shell with $s = s_0$ we have $E(s_0) = E_F$ and from eq. (13) we obtain

$$c_m(s_0) = -\frac{1}{N}\sum_{m'}' J(s_0|m, m') \tag{35}$$

and furthermore

$$u_m(s_0)^2 = v_m(s_0)^2 = \tfrac{1}{2}, \quad \varepsilon_m(s_0) = c_m(s_0).$$

In this case the difference between energies in the excited and ground superfluid states will be proportional to J.

7. Conclusion

We have considered a rather idealized model of the atomic nucleus, which is deficient in many respects. Analogous calculations may be made for other nuclear models in which the interactions of all the nucleons are taken into account. As shown in ref.[5]) for nuclear matter, account of n—p interactions together with p—p and n—n interactions leads to great complications and requires a modification of the mathematical technique.

The results obtained confirm the considerations in ref.[8,2]) regarding the physical similarity between the Fermi-systems of metals and nuclei and illustrate the fruitfulness of application of the mathematical methods developed in the theory of superconductivity [1]) for studying the properties of the atomic nucleus †.

In conclusion the author expresses his gratitude to N. N. Bogolubov for his constant interest in the present work and for valuable remarks.

† After this paper had been completely prepared for publication the author received a preprint by Belyaev [10]) who obtained some similar results by using the method described in ref.[1]) in a somewhat different manner.

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A $0^+ \rightarrow 0^+$ TRANSITION IN Ce^{140}

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Abstract: A search has been made for gamma-quanta of energy $\approx$ 1900 keV in the spectrum of La^{140} by means of a gamma-spectrometer by analysis of recoil electrons. It is shown that if such exist, their intensity is $< 0.4 \times 10^{-3}$ quantum per disintegration. It is simultaneously confirmed that the spectrum of conversion electrons contains intense conversion lines corresponding to a transition energy of 1902 keV.

The lower limit of the conversion coefficient of this transition proved equal to 0.38. Such a large value of this limit indicates that it is either a transition of high multipole order ($l > 10$), or a $0^+ \rightarrow 0^+$ transition in which gamma quanta of the given energy are entirely absent. The first assumption may be discarded since in this case the lifetime of the 1900 keV state should be very great ($> 10^{10}$ years), whereas the intensity of the corresponding conversion line falls off with a halflife of $\approx$ 40 hours. We are thus dealing here with a new $0^+ \rightarrow 0^+$ transition.

The paper also discusses the characteristics of several other transitions in Ce^{140}.

In refs.[1,2]) the existence of a conversion line, corresponding to a transition of 1909 keV energy in Ce^{140} (the average value for the two papers), is reported. According to ref.[1,2]), the intensity of this conversion line is comparatively great, comprising approximately $\frac{1}{3}$ the intensity of the K-1596 keV line. On the other hand, quanta of energy 1596 keV are emitted in practically every disintegration of La^{140}.

The γ-spectrum of La^{140} was studied by us in refs.[3,4]). In ref.[4]) we reported that if γ-rays of energy $\approx$ 1900 keV are emitted in the disintegration of La^{140}, their intensity is less than 0.0015 quantum per disintegration.

We advanced the assumption that the conversion electrons K-1909 correspond to a transition in Ce^{140} between the 1909 keV excited state with quantum characteristic 0^+ and the ground state (also 0^+).

In order to test this assumption we made a careful study of the γ-spectrum of La^{140} in the vicinity of 1900 keV. The measurements were performed on a ritron type γ-spectrometer [5]). Our source was lanthanum oxide subjected to bombardment by thermal neutrons. The initial activity of the sample comprised 2.8 Cu.

The results of the measurements (minus the background, which in this region comprised $\approx$ 0.8 impulse per minute) are represented in fig. 1. In the scale of the figure the peak of the 1596 keV γ-line corresponds to the ordinate $\approx$ 1000. The experimental points lie on a smooth curve. No

γ-line is observed in the region 1800—2000 keV; the height of the maximum of the corresponding γ-line of intensity 10^{-3} quantum per disintegration should be equal to unity. The slight surplus of recoil electrons over the background is due to the drop in the curve to the right of the strong 1596 keV γ-line and to the left of the weak 2343 keV and 2515 keV γ-lines.

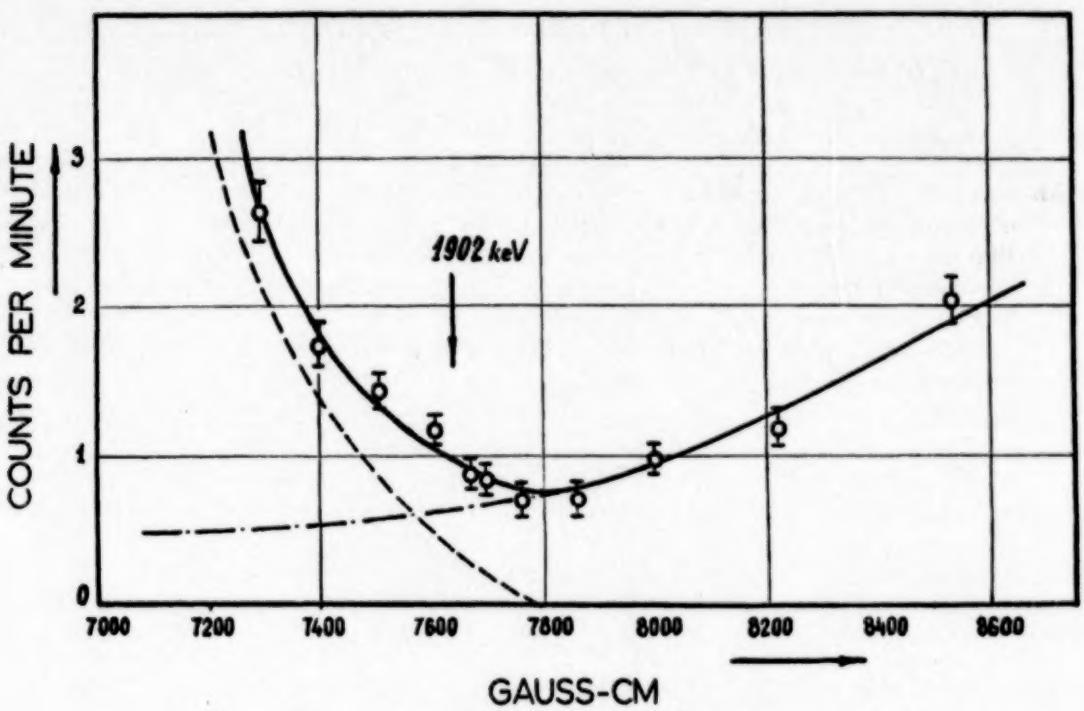

Fig. 1. Section of γ-spectrum of La^{140} in the region of 1900 keV. The dotted lines represent the "tails" of the γ-lines 1596 and 2343+2515.

In order to estimate the contribution of these lines in the energy region under investigation we made a careful study of the form of the drop in the curve to the left of the 2760 keV γ-line of Na^{24} and to the right of the 1331 keV γ-line of Co^{60}. The drops of the curve from the lines 1596 and 2343+2515 keV, as estimated from the data indicated, are depicted in fig. 1 (dotted line). Their sum corresponds to the experimental curve.

Thus, the intensity of γ-rays of $\approx$ 1900 keV, if such exist at all, comprises less than 4×10^{-4} quantum per disintegration. These data, in conjunction with the data of refs. [1] and [2]) relating to the intensity of the 1909 keV conversion line are sufficient to justify the conclusion that we are dealing here with an $0^+ \rightarrow 0^+$ transition. In the references mentioned, however, a study was made of the entire complex spectrum of conversion electrons of La^{140}, no special attention being paid to the line of interest here. Hence, in order to confirm the existence of the K-1909 line and to determine its intensity more exactly we carried out an investigation of the high-energy part of the spectrum of conversion electrons of La^{140} utilizing two caetron type

β-spectrometers which were kindly placed at our disposal for this purpose by A. A. Bashilov (Leningrad State University) and K. Y. Gromov (Radium Institute of the USSR Academy of Sciences). If both cases our source was La^{140} separated from a solution containing Ba^{140}. The results of the experiments are depicted in fig. 2. The existence of the K- and L-1909 lines is beyond doubt. The average transition energy comprises 1902 keV. The ratio of the intensities is $K_{1902}/K_{1596} = 0.220$ (averaged over all the series of experiments).

Utilizing the upper limit of the intensity of the 1902 keV γ-line (4×10^{-4}) obtained in our experiments and the known [6]) conversion coefficient for the 1596 keV transition $\alpha_K = 6.9\times10^{-4}$ (an E2 transition), we can determine the lower limit of the conversion coefficient for the 1902 keV transition. We find $\alpha_{K-1902} > 0.38$.

One might attempt to explain such a high conversion coefficient by a high multipole order (> 10). But in this case we should have an isomer level with a lifetime $> 10^{10}$ years.

But the counting rate on the K-1902 line takes off with a period $T \approx 38$ hours, which is close to the halflife of La^{140} ($T = 40.2$ hours). Hence, the only possible conclusion is that the 1902 keV transition in Ce^{140} is a new $0^+ \to 0^+$ transition. The ratio α_K/α_L for this transition is 6.33.

We simultaneously sought and discovered K- and L-conversion lines corresponding to the 2343 and 2890 keV γ-lines found by us in ref.[3]) and to the 2515 keV γ-lines. The intensities of the K-lines are respectively 4.5×10^{-3}, 0.22×10^{-3} and 21×10^{-3} of the intensity of K-1596, while $(\alpha_K/\alpha_L)_{2343} = 6.56$ and $(\alpha_K/\alpha_L)_{2515} = 6.28$. The table presents the experimental and theoretical[6]) conversion coefficients on the K-shell for the transitions indicated. The accuracy of the experimental values is estimated at 20 %.

TABLE 1

Determination of the multipole order of the 2350, 2525 and 2910 keV transitions in Ce^{140}

Transition energy: average value for ref.[3]) and the present paper $h\nu$(keV)	$\alpha_{K\,exp.} \times 10^4$	ref. [6]) $\alpha_{K\,theor.} \times 10^4$						Multipole order
		E1	E2	E3	E4	M1	M2	
2350	3.94	1.66	3.39	5.82	9.12	4.03	8.32	M1, E2
2525	3.91	1.45	2.95	5.01	7.76	3.46	7.08	M1
2910	1.87	1.07	2.24	3.63	5.50	2.45	4.79	E2, M1 E2+M1

It should be observed that the transitions discussed in the present paper offer direct proof of the existence in Ce^{140} of excited states of energies 1902, 2350, 2525 and 2910 keV.

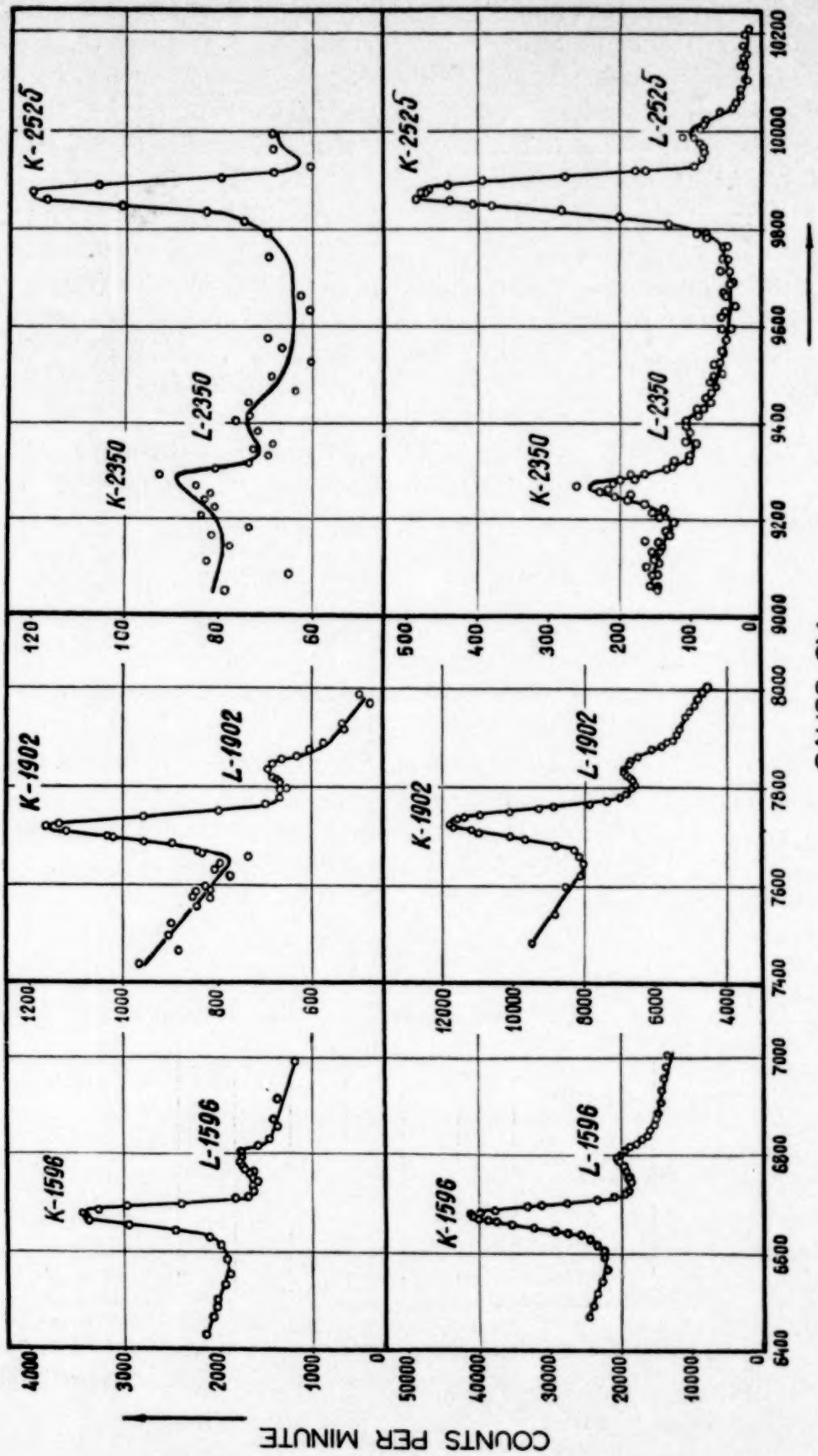

Fig. 2. High rigidity region of the spectrum (10 200 > $H\rho$ > 6400) of conversion electrons of La^{138}. Below: spectrum obtained on the caetron of Leningrad State University; above: spectrum obtained on the caetron of the Radium Institute of the USSR Academy of Sciences.

The γ-spectrometer was utilized to determine the period of the decline of intensity of the 868 and 2350 keV γ-lines detected in ref.[3]). It proved equal

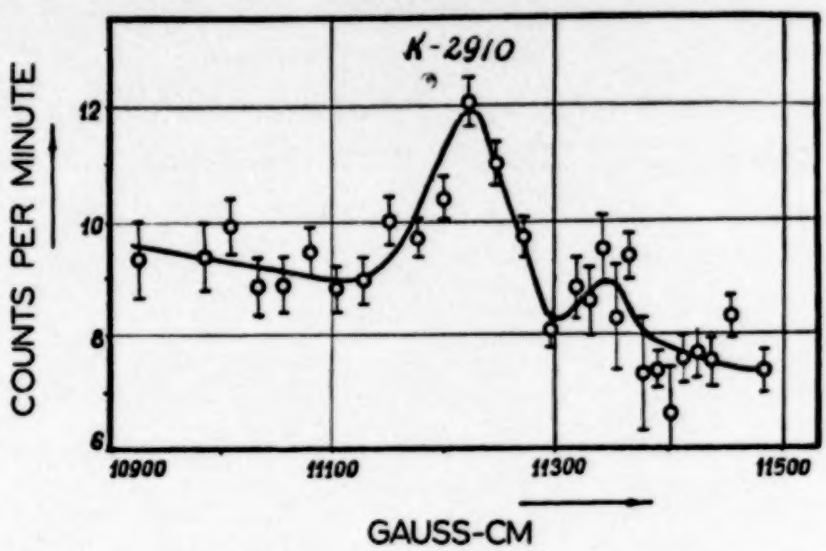

Fig. 3. Spectrum of conversion electrons of La^{140} in the region $11\,500 > Hp > 10\,900$, obtained on the caetron of Leningrad State University.

to ≈ 40 hours. This circumstance provides additional support of the assumption that the lines in question should be attributed to La^{140}.

We wish to express our gratitude to A. A. Bashilov and K. Y. Gromov for allowing us to check our experiments on their apparatus, to N. D. Novosiltseva for the chemical separation of La^{140} from a solution of barium, and to A. G. Dmitriev and V. V. Iljin for help in the measurements.

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DENSITY OF LOW-LYING PARTICLE LEVELS IN ODD-MASS NUCLEI IN THE REGIONS OF LARGE DEFORMATION

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Abstract: From experimental level schemes the mean densities of low-lying levels formed by excitation of single particles are found for nuclei with odd Z and odd N in the regions $A = 153-185$ and $A = 231-245$.

These level densities are nearly twice as high as those for a Fermi gas of nucleons where the interaction is taken into account only by using a constant effective mass $m^* = 0.7\,m$.

Such an increase in the density of the low-lying levels will be caused by a residual interaction between the nucleons at the top of the Fermi distribution. This effect may also be interpreted in terms of a pairing interaction between the nucleons.

1. Introduction

The increasing amount of data on low-lying levels in nuclei and their interpretation in terms of particle or collective motion, using the shell and the collective models, makes it possible to determine the mean level density with some accuracy. By comparing it with for instance a Fermi gas model of the nucleus one may hope to obtain some indication of the interaction between the nucleons at the top of the Fermi gas.

2. Level Data and Level Densities

One would expect to obtain the best value for the mean level density from the data on the deformed heavier nuclei where the grouping of the levels into shells and sub-shells is smeared out.

Data on low-lying levels of odd-A nuclei in the regions of large deformation $A = 153-185$ and $A = 231-245$ have been collected and analysed by Mottelson and Nilsson [1]). They were able to divide up the spectra into rotational bands based on single-particle levels and associate these single-particle levels with the levels calculated by Nilsson for a deformed potential [2]).

Such single-particle excitation levels are formed in two ways, either by "particle excitation" or by "hole excitation". In the former case the last single particle is raised to a higher level. In the latter case a pair is broken, and one of the paired particles is lifted to the top level, where it forms a new pair with the single particle which is already there.

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The order of the levels in Nilsson's scheme might not be strictly correct and may even change a little from nucleus to nucleus. If so, a definite designation of an excited level as a "hole" level or a "particle" level is difficult to make.

Therefore, half of the excited single-particle levels are assumed to be due to hole and half to particle excitation. Deviations from this assumption will lead to higher values for the level densities.

The data and results thus obtained are collected in tables 1—4.

3. Discussion of Results

The columns of tables 1—4 give the nucleus, the number of the odd particles (protons or neutrons), the energy, spin and parity of the levels assigned to particle or hole excitation, the number of excited levels, and the energy range for these levels (the latter is taken as the sum of the last two level energies, assuming one of these to be due to particle excitation and the other to hole formation); the last column gives the level densities calculated from the two preceding columns. The errors assigned to the mean values are the mean squares of the deviations of the values for each nucleus.

The systematic errors are discussed below. They will most probably make the values obtained here for the level densities of the nuclei in the rare-earth region lower than the true values.

A comparison of the levels found with those which should occur as low-lying levels for the different nuclei according to Nilsson's scheme shows that almost all expected levels are found. Only in Lu^{177} a level of spin $\frac{1}{2}+$ and in Re^{185} levels of spin $\frac{9}{2}-$ and $\frac{1}{2}-$ seem to be missing. If these were included, the level density for the odd-proton nuclei about $A = 170$ would increase by approximately 10 %.

In the region $A = 231-245$ the levels found cover only a small part of the whole shell, and local fluctuations in the level density may strongly influence the average value. This is the cause of the high density of proton levels just above the proton number 90. Also in the odd-neutron nuclei with a neutron number just above 90 the same levels occur, and the same high level density is observed. But here their effect is compensated by the levels higher up in the shell.

In the case of hole excitation, difference in pairing energy between initial and final pairs will have great influence on the distribution of levels in the different nuclei. But as the difference may be either positive or negative, the net effect will be small for the mean level density.

The most important systematic errors in the region $A = 170$ are thus due to missing levels and to the assumption of half hole and half particle excitation. Both types of error will make the obtained values smaller than

TABLE 1

Density of proton levels in odd-proton nuclei in the region $A = 153$—185

Nucleus	Z	Particle levels		No. of excited levels Δn	Energy range ΔE (keV)	Level density $d = \Delta n/\Delta E$ (levels · MeV^{-1})
		Energy (keV)	Spin and parity			
Eu^{153}	63	0	$\frac{5}{2}+$	3	813	3.69
		98	$\frac{5}{2}-$			
		103	$\frac{3}{2}+$			
		710	$\frac{1}{2}+$			
Tb^{157}	65	0	$\frac{3}{2}+$	1	327	3.06
		327	$(\frac{5}{2}-)$			
Tb^{159}	65	0	$\frac{3}{2}+$	1	364	2.75
		364	$(\frac{5}{2}-)$			
Tb^{161}	65	0	$\frac{3}{2}+$	4	1065	3.76
		316	$\frac{5}{2}+$			
		418	$\frac{7}{2}-$			
		482	$(\frac{5}{2}-)$			
		583	$(\frac{1}{2}+)$			
Ho^{163}	67	0	$\frac{7}{2}-$	1	305	3.28
		305	$\frac{1}{2}+$			
Ho^{165}	67	0	$\frac{7}{2}-$	2	876	2.28
		361	$(\frac{3}{2}+)$			
		515	$(\frac{1}{2}+)$			
Tm^{169}	69	0	$\frac{1}{2}+$	2	695	2.88
		316	$\frac{7}{2}+$			
		379	$\frac{7}{2}-$			
Tm^{171}	69	0	$\frac{1}{2}+$	4	1609	2.49
		425	$\frac{7}{2}-$			
		636	$\frac{7}{2}+$			
		688	$(\frac{3}{2}+)$			
		921	$(\frac{5}{2}+)$			
Lu^{173}	71	0	$(\frac{9}{2}-)$	1	123	8.13
		123	$(\frac{7}{2}+)$			
Lu^{175}	71	0	$\frac{7}{2}+$	3	901	3.33
		343	$\frac{5}{2}+$			
		396	$\frac{9}{2}-$			
		505	$(\frac{1}{2}+)$			
Lu^{177}	71	0	$\frac{7}{2}+$	3	1520	1.97
		147	$\frac{9}{2}-$			
		290	$(\frac{5}{2}+)$			
		1230	$\frac{7}{2}-$			
Ta^{179}	73	0	$\frac{7}{2}+$	1	30	33.33
		30	$\frac{9}{2}-$			
Ta^{181}	73	0	$\frac{7}{2}+$	4	1570	2.55
		152	$\frac{9}{2}-$			
		480	$\frac{5}{2}+$			
		612	$\frac{1}{2}+$			
		958	$(\frac{1}{2}-)$			
Re^{183}	75	0	$\frac{5}{2}+$	2	1599	1.25
		496	$\frac{9}{2}-$			
		1103	$\frac{1}{2}+$			
Re^{185}	75	0	$\frac{5}{2}+$	2	1518	1.32
		646	$\frac{1}{2}+$			
		872	$\frac{1}{2}+$			
Estimate of average density				34	13315	2.55 ± 0.47

TABLE 2

Density of neutron levels in odd-neutron nuclei in the region $A = 155-185$

Nucleus	N	Particle levels		No. of excited levels Δn	Energy range ΔE (keV)	Level density $d = \Delta n/\Delta E$ (levels · MeV^{-1})
		Energy (keV)	Spin and parity			
Gd^{155}	91	0	$\frac{3}{2}-$	2	192	10.41
		87	$\frac{1}{2}+$			
		105	$\frac{5}{2}+$			
Gd^{157}	93	0	$\frac{3}{2}-$	1	200	5.00
		200	$(\frac{1}{2}+)$			
Dy^{161}	95	0	$\frac{5}{2}+$	2	101	19.80
		26	$\frac{3}{2}-$			
		75	$\frac{1}{2}-$			
Er^{165}	97	0	$\frac{5}{2}-$	2	600	3.33
		243	$\frac{1}{2}-$			
		357	$\frac{3}{2}-$			
Dy^{165}	99	0	$\frac{7}{2}+$	1	108	9.25
		108	$\frac{1}{2}-$			
Er^{167}	99	0	$\frac{7}{2}+$	3	1050	2.86
		208	$\frac{1}{2}-$			
		350	$(\frac{5}{2}-)$			
		700	$\frac{9}{2}-$			
Yb^{169}	99	0	$\frac{7}{2}+$	1	200	5.00
		200	$(\frac{1}{2}-)$			
Yb^{171}	101	0	$\frac{1}{2}-$	2	206	9.70
		75	$\frac{7}{2}+$			
		131	$(\frac{7}{2}-)$			
Yb^{173}	103	0	$\frac{5}{2}-$	1	351	2.85
		351	$\frac{7}{2}+$			
Hf^{177}	105	0	$\frac{7}{2}-$	3	1818	1.65
		326	$\frac{9}{2}+$			
		748	$(\frac{5}{2}-)$			
		1070	$(\frac{7}{2}-)$			
W^{179}	105	0	$\frac{7}{2}-$	1	222	4.50
		222	$(\frac{1}{2}-)$			
Hf^{179}	107	0	$\frac{9}{2}+$	2	590	3.39
		215	$\frac{7}{2}-$			
		375	$\frac{1}{2}-$			
W^{181}	107	0	$\frac{9}{2}+$	3	1438	2.08
		366	$\frac{5}{2}-$			
		515	$(\frac{1}{2}-)$			
		923	$(\frac{3}{2}-)$			
Os^{183}	107	0	$\frac{9}{2}+$	1	171	5.85
		171	$\frac{1}{2}-$			
W^{183}	109	0	$\frac{1}{2}-$	2	662	3.02
		209	$\frac{3}{2}-$			
		453	$\frac{7}{2}-$			
W^{185}	111	0	$\frac{3}{2}-$	1	87	11.50
		87	$(\frac{7}{2}-)$			
Estimate of average density				28	7996	3.50 ± 0.74

TABLE 3

Density of proton levels in odd-proton nuclei in the region $A = 231-245$

Nucleus	Z	Particle levels		No. of excited levels Δn	Energy range ΔE (keV)	Level density $d = \Delta n/\Delta E$ (levels · MeV^{-1})
		Energy (keV)	Spin and parity			
Pa^{231}	91	0	$\frac{3}{2}-$	4	450	8.90
		59	$\frac{5}{2}-$			
		85	$\frac{5}{2}+$			
		170	$\frac{3}{2}+$			
		280	$\frac{7}{2}+$			
Pa^{233}	91	0	$\frac{3}{2}-$	2	226	8.85
		86	$\frac{5}{2}+$			
		140	$(\frac{3}{2}-)$			
Np^{235}	93	0	$\frac{5}{2}+$	1	48	20.83
		48	$\frac{5}{2}-$			
Np^{237}	93	0	$\frac{5}{2}+$	3	599	5.00
		60	$\frac{5}{2}-$			
		267	$\frac{3}{2}-$			
		332				
Np^{239}	93	0	$\frac{5}{2}+$	1	74	13.50
		74	$\frac{5}{2}-$			
Am^{239}	95	0	$\frac{5}{2}-$	2	522	3.84
		192	$\frac{3}{2}-$			
		330	$\frac{1}{2}+$			
Am^{241}	95	0	$\frac{5}{2}-$	3	1120	2.68
		206	$(\frac{5}{2}+)$			
		480	$\frac{3}{2}-$			
		640				
Am^{245}	95	0	$\frac{5}{2}-$	2	430	4.65
		40	$\frac{5}{2}+$			
		390	$\frac{7}{2}+$			
Estimate of average density				18	3469	5.20 ± 1.19

the true values. The results here agree with the single-particle level distances found from the low-energy neutron capture cross-sections [3]).

4. Fermi Gas Model and Effective Mass

As a model of the nucleus we take a Fermi gas of protons and neutrons in a sphere of radius R. The interaction between the particles is taken into account by using an effective mass m^* which may vary with the state α in which the nucleon i is. We write for the energy of this state

$$E_\alpha = \left\langle \frac{1}{2m} p_i^2 + \sum_j V_{ij} \right\rangle_\alpha = \frac{1}{2m^*} \langle p_i^2 \rangle_\alpha$$

TABLE 4

Density of neutron levels in odd-neutron nuclei in the region $A = 231-245$

Nucleus	N	Particle levels		No. of excited levels Δn	Energy range ΔE (keV)	Level density $d = \Delta n/\Delta E$ (levels · MeV^{-1})
		Energy (keV)	Spin and parity			
Th^{231}	141	0	$\frac{5}{2}+$	2	575	3.48
		185	$\frac{5}{2}-$			
		390	$\frac{7}{2}-$			
U^{233}	141	0	$\frac{5}{2}+$	3	713	4.20
		295	$\frac{7}{2}-$			
		313	$\frac{3}{2}+$			
		400	$\frac{1}{2}+$			
U^{235}	143	0	$\frac{7}{2}-$	2	172	11.62
		0.1	$\frac{1}{2}+$			
		172	$(\frac{5}{2}+)$			
Pu^{237}	143	0	$\frac{7}{2}-$	1	145	6.90
		145	$\frac{1}{2}+$			
U^{237}	145	0	$\frac{1}{2}+$	1	145	6.90
		145	$\frac{5}{2}+$			
Pu^{239}	145	0	$\frac{1}{2}+$	3	904	3.32
		286	$\frac{5}{2}+$			
		392	$\frac{7}{2}-$			
		512	$\frac{5}{2}+$			
Pu^{241}	147	0	$\frac{5}{2}+$	1	172	5.81
		172	$\frac{7}{2}+$			
Cm^{245}	149	0	$\frac{7}{2}+$	3	1026	2.92
		257	$\frac{5}{2}+$			
		394	$\frac{9}{2}-$			
		632	$\frac{1}{2}+$			
Estimate of average density				16	3852	4.15 ± 0.72

with

$$\frac{1}{2m^*} = \frac{1}{2m} + \frac{\langle \sum V_{ij} \rangle_\alpha}{\langle p_i^2 \rangle_\alpha}.$$

If some of the interaction is taken into account by using a potential well, the effective mass needed will be smaller:

$$E_\alpha = \frac{1}{2m^*} \langle p_i^2 \rangle_\alpha - \langle V \rangle_\alpha$$

with

$$\frac{1}{2m^*} = \frac{1}{2m} + \frac{\langle \sum V_{ij} \rangle_\alpha}{\langle p_i^2 \rangle_\alpha} + \frac{\langle V \rangle_\alpha}{\langle p_i^2 \rangle_\alpha}.$$

From the number of cells with wave number k in phase space

$$\mathrm{d}n = \frac{2}{3\pi} R^3 k^2 \mathrm{d}k \qquad (1)$$

and the connection between wave number and energy above the bottom of the potential,

$$k^2 = \frac{2m^* E}{\hbar^2}, \tag{2}$$

the level density is found to be

$$d = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{1}{3\pi}\left(\frac{2m^* ER^2}{\hbar^2}\right)^{\frac{1}{2}}\left(\frac{1}{E} + \frac{1}{m^*}\frac{\mathrm{d}m^*}{\mathrm{d}E}\right), \tag{3}$$

and the number of levels up to the energy E is

$$n = \frac{2}{9\pi}\left(\frac{2m^* ER^2}{\hbar^2}\right)^{\frac{3}{2}}. \tag{4}$$

Each level here may contain 2 protons and 2 neutrons. From (3) and (4) the relative increase of the effective mass at the top of the Fermi distribution is found to be

$$\frac{1}{m^*}\frac{\mathrm{d}m^*}{\mathrm{d}E} = \frac{d - \frac{3}{2}\left(\frac{2}{9\pi}\right)^{\frac{1}{3}}\frac{2m^* R^2}{\hbar^2}n^{\frac{1}{3}}}{\frac{3}{2}n} = \frac{d - d_0\frac{m^*}{m}}{\frac{3}{2}n}, \tag{5}$$

where d_0 is the level density in a Fermi gas of particles with a mass equal to the free nucleon mass.

The energy of the nucleons at the top of the Fermi distribution is

$$E_F = \left(\frac{9\pi}{2}\right)^{\frac{2}{3}}\frac{\hbar^2}{2m^* R^2}n^{\frac{2}{3}}. \tag{6}$$

The values obtained for these quantities are given in columns 4 and 5 of table 5. The radius used here is [4]) $R_0 = (0.8 + 1.23\, A^{\frac{1}{3}}) \times 10^{-13}$ cm, and for the effective mass we have taken the fairly high value $m^* = 0.7\, m$ resulting from many determinations [5]).

TABLE 5

Relative increase of the effective mass at the top of the Fermi distribution

Nuclei	n	d (levels · MeV^{-1})	$d_0\frac{m^*}{m}$ (levels · MeV^{-1})	$\frac{1}{m^*}\frac{\mathrm{d}m^*}{\mathrm{d}E}$ (MeV^{-1})	E_F (MeV)
$A \approx 170$					
Odd Z	35	2.55	1.79	0.014	32
Odd N	50	3.50	2.02	0.022	40
$A \approx 240$					
Odd Z	46	5.20	2.33	0.044	32
Odd N	72	4.15	2.72	0.015	42

It is seen from columns 2 and 3 that the observed level densities are almost a factor 2 greater than those of a Fermi gas with constant effective mass equal to $m^* = 0.7\ m$. Column 4 shows that an increase of the effective mass of only about 2%/MeV is sufficient to press the levels together by a factor of 1.7. Some of this increase will be due to the diffuseness of the walls of the nucleus. This effect may be estimated by using an energy-dependent radius for the sphere in which the Fermi gas is enclosed:

$$R = R_m + a\frac{E}{V},$$

where R_m is the radius at the bottom of the potential well. It is adjusted so that $R = R_0$ at the top of the Fermi distribution. The wall thickness used is $a = 2.4\times 10^{-13}$ cm.

Assuming the effective mass independent of E except in a small region about the top of the Fermi distribution, the effect of the diffuse wall will be to increase the energy of the nucleons at the top by a factor $[1+(aE/RV)]^{\frac{1}{2}} \approx 1.15$ and the level density d_0 by approximately 8 %.

The diffuseness of the nuclear wall can thus account for only 10 % of the relative increase of the effective mass.

Other deviations from the Fermi gas model due to the shape of the nuclear potential will have but negligible effect on the level density.

A sharp increase of the effective mass and of the level density about the top of the Fermi distribution may, however, be caused by a residual interaction between the nucleons in a layer of the thickness 2—3 MeV about the top. Such an interaction has been considered by Bardeen *et al.*[6]) in connection with the theory of superconductivity. For a single particle at the top they obtain a level spectrum of the form

$$E_k = \sqrt{\varepsilon_0^2+\varepsilon_k^2}-\varepsilon_0.$$

Here ε_k is the energy above the top for the level k in a nucleus without residual interaction, and ε_0 is a constant which may be estimated from the gap observed in the particle level spectra of the neighbouring even nuclei. For the nuclei in the rare-earth region this gap is of the order 1.2 MeV, which gives $\varepsilon_0 = 0.6$ MeV. This increases the level density about the first excited particle level by a factor of nearly 2, which is what is needed.

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ON THE THEORY OF MULTIPLE SCATTERING

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Abstract: A method for calculation of multiple scattering curves taking into account the finite dimensions of the nucleus is presented. Experimental results pertaining to scattering of fast electrons by nuclei are used in the calculations.

Let $f(\theta_x, \theta_y, t)$ represent the angular distribution function of charged particles at a depth t g/cm². The scattering angles $\boldsymbol{\theta}$ of the particles will be assumed to be small and can therefore be represented by a vector lying in a plane perpendicular to the direction of motion. θ_x and θ_y are projections of the scattering angle on two mutually perpendicular planes passing through the initial direction of motion. For small scattering angles the distribution function obeys the kinetic equation

$$\frac{\mathrm{d}f(\theta_x, \theta_y, t)}{\mathrm{d}t} = -f(\theta_x, \theta_y, t) \int\int_{-\infty}^{+\infty} \frac{N}{A}\sigma(\theta'_x, \theta'_y)\mathrm{d}\theta'_x\mathrm{d}\theta'_y + \int\int_{-\infty}^{+\infty} \frac{N}{A}\sigma(\theta'_x, \theta'_y) f(\theta_x - \theta'_x; \theta_y - \theta'_y)\mathrm{d}\theta'_x\mathrm{d}\theta'_y. \tag{1}$$

Here N is Avogadro's number, A the atomic weight, $\sigma(\theta_x, \theta_y)$ the scattering cross section.

The distribution function satisfies the boundary condition

$$f(\theta_x, \theta_y, 0) = 2\delta(\theta_x)\delta(\theta_y) \tag{2}$$

and the normalization condition

$$\int\int_{-\infty}^{+\infty} f(\theta_x, \theta_y, t)\mathrm{d}\theta_x\mathrm{d}\theta_y = 2. \tag{3}$$

Expanding the distribution function into a double Fourier integral

$$f(\theta_x, \theta_y, t) = \int\int_{-\infty}^{+\infty} f(k_x, k_y, t)\mathrm{e}^{\mathrm{i}(k_x\theta_x + k_y\theta_y)}\,\mathrm{d}k_x\mathrm{d}k_y \tag{4}$$

we can rewrite eq. (1) as

$$\frac{df(k_x, k_y, t)}{dt} = -f(k_x, k_y)\frac{N}{A}\sigma(0, 0)+f(k_x, k_y)\frac{N}{A}\sigma(k_x, k_y) \tag{5}$$

where

$$\sigma(k_x, k_y) = \int e^{-i(k_x\theta_x+k_y\theta_y)}\sigma(\theta_x, \theta_y)d\theta_x, d\theta_y$$

and $\sigma(0, 0)$ is the total scattering cross section. The solution of eq. (1) can be immediately written down by using the solution of eq. (5) and the Fourier transformation (4)

$$f(\theta_x, \theta_y, t) = C\iint_{-\infty}^{+\infty} e^{i(k_x\theta_x+k_y\theta_y)}e^{(N/A)t[\sigma(k_x, k_y)-\sigma(0, 0)]}dk_x dk_y; \tag{6}$$

C is defined by the normalization condition: $C = 1/2\pi^2$. In the future we shall need the distribution function either for θ_x of θ_y. This is connected with the circumstance that in cloud chambers it is much easier to measure projections of the scattering angle than the spatial scattering angle itself. Therefore solution (6) should be integrated over one of the projections of the angle. Integration over θ_y yields a delta function which permits one to perform the integration with respect to k_y in the double integral (6). The final result can be represented in the form

$$f(\theta_x, t)d\theta_x$$
$$= \frac{d\theta_x}{\pi}\int_{-\infty}^{+\infty} e^{ik_x\theta_x}\exp\left\{\frac{Nt}{A}\int_{-\infty}^{+\infty} e^{ik_x\theta'_x}\sigma(\theta'_x\theta'_y)d\theta'_x d\theta'_y - \frac{Nt}{A}\sigma(0, 0)\right\}dk_x \tag{7}$$

where $f(\theta_x t) = \int_{-\infty}^{+\infty} f(\theta_x, \theta_y, t)d\theta_y$ and the normalization condition $\int_0^\infty f(\theta_x, t)$ $d\theta_x = 1$ is satisfied. The latter follows from condition (3) and from the fact that $f(\theta_x, t)$ is an even function, which is always true as $\sigma(\theta_x, \theta_y)$ is always an even function of its arguments. Solution (7) is valid for any scattering law and for small scattering angles.

We shall temporarily confine our attention to pure Coulomb scattering on a fixed nucleus. Then $\sigma(\theta_x, \theta_y)$ represents the Rutherford scattering cross section in which the atomic as well as the nuclear form factor have been taken into account:

$$\frac{Nt}{A}\sigma(\theta_x, \theta_y) = 2\chi_c^2\frac{q^2F^2}{(\theta_x^2+\theta_y^2)^2}. \tag{8}$$

We employ here the quantity

$$\chi_c^2 = \frac{4\pi Nt}{A}\frac{Z(Z+1)z^2e^4}{p^2v^2} \tag{9}$$

usually encountered in the theory of multiple scattering; Z is the charge of the scattering nucleus, z the charge of the scattered particle (assumed to be unity in the following), p and v the momentum and velocity of the scattered particle, q the atomic form factor and F the nuclear form factor.

In the Born approximation

$$q = 1 - \frac{1}{Z}\int \rho_e(r) e^{i\mathbf{k}\cdot\mathbf{r}} dv, \tag{10}$$

$$F = \frac{1}{Z}\int e^{i\mathbf{k}\cdot\mathbf{r}} \rho_{nuc}(r) dv \tag{11}$$

where ρ_e and ρ_{nuc} are respectively the electron density in the atom and proton density in the nucleus; $\mathbf{k} = \mathbf{k}_0 - \mathbf{k}_1$ is the change of the wave vector. The Born approximation is valid for $\alpha \equiv zZe^2/hv \ll 1$. The expression for q must be known for all values of α. Molière [1]) has shown how the exact formula for a screened Coulomb centre can be derived if the wave length of the impinging particle is smaller than the dimensions of the atom. The deduction involves the substitution of $q^2/(\theta_x^2+\theta_y^2)^2$ for

$$(\theta_x^2+\theta_y^2+\varphi_m^2)^{-2}$$

where

$$\varphi_n = \frac{\lambda\!\!\!^{-}}{0.885\, R_0 Z^{-\frac{1}{3}}} \sqrt{1.13+3.76\,\alpha^2}. \tag{12}$$

Here $\lambda\!\!\!^{-}$ is the wave length of the impinging particle and R_0 the radius of the hydrogen atom. It should be noted that application of the Born approximation ($\alpha \ll 1$) instead of (12) considerably distorts the multiple scattering function inasmuch as low angle scattering ($\theta \approx \varphi_m$) is a very frequent process and the uncertainty in the cross section for small angles appreciably affects the final result.

The nuclear form factor is important at angles considerably exceeding φ_m. The nuclear screening angle φ_{nuc} is given, as to order of magnitude, by $\varphi_{nuc} \approx \lambda\!\!\!^{-}/R_{nuc}$ where R_{nuc} is the nuclear radius. The nuclear form factor has been measured for a number of elements by Hofstadter [2]). For electrons of hundreds of MeV energy F^2 in (8) can be approximated by the following expression [3]):

$$F^2 = \frac{1}{\{1+k^2a^2(\theta_x^2+\theta_y^2)\}^4}. \tag{13}$$

The quantity a characterizes the scattering nucleus. Thus for lead $a = 2.36\times10^{-13}$ cm. It should be mentioned that the form factor was chosen in the analytical form (13) only for the sake of convenience of the calculations. From a physical viewpoint this analytical form is inadmissible as it leads to an exponential distribution of the nuclear density. Expression (13) should merely be considered as an interpolation of the experimental results in the electron momentum range of the order of 100 MeV/c. It seems natural to assume that expression (13) will also be true for scattering of other types of particles in the same range of variation of the momentum.

Using (8), (13) and (12) we can integrate the exponent of expression (7) over θ'_y. This yields

$$f(\theta_x)\mathrm{d}\theta_x = \frac{2}{\pi}\mathrm{d}\theta_x \int_0^\infty \cos k_x\theta_x \exp\left\{-\tfrac{2}{3}\chi_c^2 \int_0^\infty (1-\cos k_x\theta'_x)B(\theta'_x)\mathrm{d}\theta_x\right. \quad (14)$$

where

$$B(\theta_x) = \frac{3}{2(\varphi_m^2+\theta_x^2)^{\frac{1}{2}}} - \frac{12}{\varphi_{nuc}^2(\theta_x^2+\varphi_m^2)^{\frac{1}{2}}} + \frac{15x^2}{16(\theta_x^2+\varphi_{nuc}^2)^{\frac{1}{2}}} + \frac{9\varphi_{nuc}^2}{4(\theta_x^2+\varphi_{nuc}^2)^{\frac{1}{2}}} + \frac{9}{2(\theta_x^2+\varphi_{nuc}^2)^{\frac{1}{2}}} + \frac{12}{\varphi_{nuc}^2(\theta_x^2+\varphi_{nuc}^2)^{\frac{1}{2}}} \quad (15)$$

and $\varphi_{nuc} = 1/k_a$. It is easy to see that the probability of single scattering in the plane angle interval θ_x, $\theta_x+\mathrm{d}\theta_x$ after traversal of a thickness t is $\frac{1}{3}\chi_c^2$ $B(\theta_x)\mathrm{d}\theta_x$. The integral in the exponent in expression (14) can also be evaluated analytically and we finally obtain

$$f(\theta_x)\mathrm{d}\theta_x = \frac{2}{\pi}\mathrm{d}\theta_x \int_0^\infty \mathrm{d}\xi \cos\theta_x\xi \exp\left\{-\frac{\chi_c^2\xi^2}{2}\left(\tfrac{1}{2}-0.577-\ln\frac{\xi\varphi_m}{2}\right)\right.$$
$$-\frac{8\chi_c^2}{\varphi_{nuc}^2}\left[-0.577+\ln\frac{2}{\xi\varphi_{nuc}} - K_0(\xi\varphi_{nuc})\right] - \frac{13}{3}\frac{\chi_c^2}{\varphi_{nuc}^2}[1-\xi\varphi_{nuc}K_1(\xi\varphi_{nuc})] \quad (16)$$
$$\left.+\tfrac{2}{3}\chi_c^2\xi^2K_0(\xi\varphi_{nuc}) + \frac{\xi^3\varphi_{nuc}\chi_c^2}{24}K_1(\xi\varphi_{nuc})\right\};$$

$K_1(\xi\varphi_m)$ and $K_0(\xi\varphi_m)$ have both been expanded into a series, since it is the region $\xi\varphi_m \ll 1$ which is important. For $\varphi_{nuc} \gg \chi_c$ we arrive at the expression obtained by Molière for the distribution function of a point screened Coulomb centre:

$$f(\theta_x)\mathrm{d}\theta_x = \frac{2}{\pi}\mathrm{d}\theta_x \int_0^\infty \mathrm{d}\xi \cos\theta_x\xi \exp\left\{-\frac{\chi_c^2\xi^2}{2}\left[\tfrac{1}{2}-0.577-\ln\frac{\varphi_m\xi}{2}\right]\right\}. \quad (17)$$

We introduce the Molière parameter B by

$$B-\ln B = 2\left(\tfrac{1}{2}-0.577-\ln\frac{\varphi_m}{\chi_c}\right)$$

and measure angles in units of $\chi_c\sqrt{B}$. Denoting $\varphi = \theta/\chi_c\sqrt{B}$ we then obtain

$$f(\theta_x)\mathrm{d}\theta_x = \frac{2}{\pi}\mathrm{d}\varphi \int_0^\infty \mathrm{d}y \cos\varphi y \exp\left\{-\frac{y^2}{4}\left[1-\frac{1}{B}\ln\frac{y^2}{4}\right]\right\}. \quad (18)$$

We have obtained Molière's expression which can be expanded in terms of $1/B$. The values of the coefficients are tabulated in Molière's paper [1]). It should be noted that the curves $f(\theta_x)$ expressed in terms of φ have a universal form and depend only on a single parameter. Expression (16) is thus a solution of the problem of determining the distribution function taking into account the finiteness of the nuclear dimensions.

Expression (16) can be put into a form convenient for numerical computations. Introducing the variables

$$\frac{\theta_x}{\varphi_{\text{nuc}}} = \theta'_x, \qquad \xi\varphi_{\text{nuc}} = y, \tag{19}$$

that is, measuring the scattering angles in units of the nuclear angle, we obtain

$$f(\theta_x)\mathrm{d}\theta_x = \frac{2\mathrm{d}\theta'_x}{\pi}\int_0^\infty \mathrm{d}y \cos\theta'_x y \exp\left\{-\frac{\chi_c^2}{\varphi_{\text{nuc}}^2}\left[\frac{y^2}{2}\left(\tfrac{1}{2}-0.577-\ln y\frac{\varphi_m}{2\varphi_{\text{nuc}}}\right)\right.\right. \tag{20}$$

$$\left.\left.+8\left(-0.577+\ln\frac{2}{y}-K_0(y)\right)+\tfrac{13}{3}\left(1-yK_1(y)\right)-\tfrac{2}{3}y^2K_0(y)-\frac{y^3}{24}K_1(y)\right\}.$$

The result of the integration depends on the two parameters

$$\frac{\chi_c}{\varphi_{\text{nuc}}} = \frac{vh}{aZe^2}\frac{1}{\sqrt{\left(4\pi\frac{N}{A}t\right)}}, \tag{21}$$

$$\frac{\varphi_m}{\varphi_{\text{nuc}}} = \frac{1.14 m_e c Z^{\frac{1}{3}} a}{137h}\left[1.13+3.76\left(\frac{Ze^2}{hv}\right)^2\right]^{\frac{1}{2}}. \tag{22}$$

It can be seen from these expressions that both parameters depend only on the particle velocity v. This means that the same curves for given values of t, A and Z can be used for different types of particles possessing the same velocities.

Formula (20) was used to compute the scattering curves in 4.5 g/cm² and 8.5 g/cm² lead plates for velocities $0.61c$ (μ-meson momentum 80 MeV/c); $0.73\,c$ ($p_\mu = 110$ MeV/c), $0.78\,c$($p_\mu = 130$ MeV/c), $0.85\,c$ ($p_\mu = 170$ MeV/c). These curves were employed in ref.[4,5]) in an analysis of the experimental results on π- and μ-meson and also proton scattering. It might be mentioned that usually values of the angle expressed in units of the characteristic multiple scattering angle $\varphi = \theta_x/\chi_c\sqrt{B}$ are plotted along the abscissa axis. We, however, have obtained curves (eq. (20)) in which the angles are expressed in units of the "nuclear angle" $\theta'_x = \theta_x/\varphi_{\text{nuc}}$.

If we rewrite expression (20) in the form

$$f(\theta_x)\mathrm{d}\theta_x = f_1(\theta'_x)\mathrm{d}\theta'_x = f_2(\varphi)\mathrm{d}\varphi$$

it should be clear that a change to the variable φ is not difficult. It is important to note that the change in the scale which must be performed depends only on the particle velocity $f_1\sqrt{(B)}/\varphi_{\text{nuc}}$ (but not on the mass) inasmuch as B is a function only of the velocity. Therefore, after recalculation the curves can be applied to different types of particles possessing the same velocity. For the curves obtained in this way the mean values of the scattering angles and the mean squares were found by numerical integration. For illustration we present in table 1 some results for a 8.5 g lead plate

TABLE 1

v/c	0.61	0.78	0.85	point nucleus
$\bar{\varphi}$	0.53 0.52	0.54 0.537	0.56 0.54	0.60
$\sqrt{\bar{\varphi}^2}$	0.66 0.65	0.69 0.67	0.71 0.66	0.77
$(\chi_c\sqrt{B})_\mu$	34.6°	20.2°	15.5°	
A	532	550	561	611
B	672	694	711	776

In some squares two values are presented. The upper values are the calculated quantities for a cut-off at $\varphi = 2.4$ and the lower for a cut-off at $\varphi = 2$. The values for a point nucleus were computed from Molière's distribution curve, the Molière formula

$$\bar{\varphi} = \frac{2}{\pi}\bar{\theta} = \frac{1}{\pi}\left\{1 + \frac{0.982}{B} - \frac{0.117}{B^2}\right\}$$

being employed in the calculations (in this formula $\bar{\theta}$ is the mean absolute value of the spatial scattering angle). The same procedure was employed for calculation of $\bar{\varphi}^2$ for a point nucleus but the cut-off was performed at $\varphi = 4$. The values of $\chi_c\sqrt{B}$ for μ-mesons are given in the fourth row. The figures in the table must be multiplied by this quantity in order to obtain the results in degrees.

The theory presented here can be applied to determine the mass from values of the scattering angle and range measured in a cloud chamber. Following Olbert we introduce the quantity $\eta = \theta R^\alpha$ (22), where R is the range, θ the scattering angle in radians, and α is defined by the following empirical equation which relates the range and quantity $\Pi = pv$:

$$\frac{R}{mc^2} = A_2\left(\frac{\Pi}{mc^2}\right)^{1/\alpha}; \tag{23}$$

m is the particle mass and A is a constant which in our case equals 0.32 g/cm² · MeV. The preceding relationship is valid in the range $0.05 \leqq \Pi/mc^2 \leqq 2$. It is easy to see that the distribution function of the quantity η differs from the distribution function of the scattering angles φ only in respect to scale. The values of $\bar{\eta}$ and $\bar{\eta}^2$ are connected with those of $\bar{\varphi}$ and $\bar{\varphi}^2$ by the relations

$$\eta = \bar{\varphi}\left(\chi_c\sqrt{B}\frac{pv}{m_e c^2}\right)(A_2 m_e c^2)^\alpha\left(\frac{m_e}{m}\right)^{1-\alpha} = A\left(\frac{m_e}{m}\right)^{1-\alpha}, \tag{24}$$

$$\sqrt{\bar{\eta}^2} = \sqrt{\bar{\varphi}^2}\left(\chi_c\sqrt{B}\frac{pv}{m_e c^2}\right)(A_z m_e c^2)^{\alpha}\left(\frac{m_e}{m}\right)^{1-\alpha} = B\left(\frac{m_e}{m}\right)^{1-\alpha}; \qquad (25)$$

m_e is the electron mass. The mass m of the unknown particle can be determined from these formulas inasmuch as the dependence on pv drops out. Values of the coefficients (factor before $(m_e/m)^{1-\alpha}$), calculated for our case are presented in the table. The values refer to a cut-off parameter $\varphi = 2.4$. Values of A and B for another cut-off parameter can be found if the corresponding values of $\bar{\varphi}$ and $\bar{\varphi}^2$ are known for the given cut-off parameter (the values in the table correspond to $\varphi_{max} = 2.4$ and $\varphi_{max} = 2$). It follows from the foregoing that the results do not strongly depend on the cut-off parameter. Details regarding the method of mass determination can be found in ref.[6]). However, the multiple scattering theory with account of the finite nuclear dimensions developed in this paper is a poor description of the true state of affairs.

In conclusion some remarks regarding the limits of applicability of the formulas derived here should be made.

First of all, we shall consider scattering of particles which in addition to Coulomb interaction possess nuclear interaction as well (π-mesons, protons, etc.). In this case the scattering cross sections which take into account nuclear interaction should be inserted in formula (7) instead of $\sigma(\theta'_x, \theta'_y)$ and $\sigma(0,0)$. It may be noted that nuclear scattering is of a diffractional nature (concentrated predominantly at angles of the order of $\lambda\!\!\!^{-}/R_{nuc}$) and that the nuclear scattering cross section at these angles may comprise a significant fraction of the total differential cross section (it should be mentioned that in the small angle region the differential scattering cross section may be considerably larger than the geometrical cross section). This can be seen from a rough estimation by comparing Coulomb scattering $d\sigma_{Coul} = 8\pi(Z_e^2/pv)(d\theta/\theta^3)F^2$ and diffractional scattering (on an absolutely black nucleus)

$$d\sigma_{diff} = R_{nuc}^2|I_1(\theta R/\lambda\!\!\!^{-})/\theta|^2 d\Omega.$$

It can immediately be seen from the formulas that at small angles the nuclear cross section is negligible compared with the Coulomb whereas the opposite is true at large angles; at some angle θ_1 the cross sections are approximately equal. The point of interest is to ascertain just how this additional (nuclear) scattering affects the multiple scattering curve. If θ_1 turns out to be much smaller than the characteristic multiple scattering angle $\chi_c\sqrt{B}$ (which in a first approximation equals the mean square angle), the multiple scattering curve based exclusively on Coulomb interaction should be greatly affected; if, however, $\theta_1 \gtrsim \chi_c\sqrt{B}$, only the distribution tail will be affected.

The real state of affairs is much more complicated and determination of the multiple scattering curve based on the exact differential scattering cross

section is made difficult because of the absence of sufficient experimental and theoretical data relating to the single scattering differential cross section. Nevertheless some conclusions regarding multiple scattering of charged nuclear-active particles can be drawn [5]).

A number of estimations of μ-meson scattering have been made. Inelastic interactions leading to excitation of the nucleus and imitating elastic scattering processes have been considered. Moreover, the probability of the μ-meson possessing an anomalous magnetic moment has been investigated, estimations of the approximations made in the kinetic equation (e.g., the small angle approximation) have been analyzed and finally momentum losses in the plates and other corrections have been considered. We shall not examine the relevant estimations here as a number of papers devoted to these questions have appeared during recent years [8,9]).

The author wishes to thank F. I. Strizhevskaya for calculation of many of the multiple scattering curves and A. I. Alikhanian, F. A. Arutyunian and also V. G. Kirillov-Ugriumov and M. Daion for stimulating discussions.

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ELECTRIC DIPOLE MOMENT ASSOCIATED WITH OCTUPOLE VIBRATIONS OF A SPHEROIDAL NUCLEUS

(Additional remarks)

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Received 31 October 1958

It has been called to our attention that the expression (5) in the article with the above title [1]) does not agree with the corresponding expression given by V. Strutinski [2]). On rechecking our derivation, we find that the effect of the centre of mass shift, of order $\beta_2\beta_3$, has not been correctly evaluated and leads to a reduction of the transition dipole moment by a factor $\frac{5}{12}$. The expression (5) should thus read

$$D = -0.0065 \cdot \frac{e^2}{R_0}\frac{A}{C_1} \cdot Z\beta_2\beta_3 eR_0, \qquad (1)$$

which is then expected to represent the dipole moment arising from the distortion of the nuclear shape, assuming a uniform neutron-proton distribution for the spherical nucleus.

In addition to the polarization effects associated with the nuclear deformation, Strutinski [2]) considers the polarization produced by the total nuclear charge (the monopole field), which tends to push the protons towards the nuclear surface. Assuming that the sum of the neutron and proton densities is a constant inside a sharply defined surface, and that the difference in the two densities is a consequence of the electric polarization effect, Strutinski finds an additional contribution to the dipole moment, which is of opposite sign to (1) and larger by a factor $\frac{7}{3}$. His expression for the net moment, if written in the notation of ref.[1]), has thus a numerical coefficient of $+0.0086$.

Thus, one sees that the estimate of the nuclear dipole moment is rather sensitive to deviations from a uniform proton-neutron density distribution in the spherical nucleus (monopole effect). However, it would appear that there is considerable uncertainty in the estimate of these effects based on the above-mentioned simple model. In fact, the difference in the total number of neutrons and protons, together with the diffuseness of the surface, is expect-

ed to have an important influence on the radial variation of the neutron and proton densities and might even change the sign of the variation from that resulting from the electric polarization effect [3]).

The uncertainty associated with estimates of the monopole effect and the sensitivity of the dipole moment to this effect makes it difficult at the present time to give a quantitative theoretical estimate of the dipole moment. Even the sign may be in doubt, and an experimental determination of this sign would thus be of value. In principle this could be obtained from a study of the interference between E1 and E3 decays, e.g. of a (1−) vibrational state to the (2+) member of the ground state band.

We are indebted to Dr. Strutinski for enlightening discussions of these problems.

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ON ANGULAR DISTRIBUTIONS IN COMPOUND NUCLEUS PROCESSES

(ADDENDUM)

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Received 11 November 1958

In the article with the above title [1]), the following considerations should be added on p. 289 after the establishment of the conditions (21) and (23):

If the condition (21) is violated, the expansion (13) still remains valid for a description of the angular distribution at not too small angles. In this case, i.e. when the incoming angular momentum is large compared with the most probable spin of the residual nucleus, there exists a limiting form for the angular distribution [2]). It is obtained either from eq. (13) for $\alpha_f I \to \infty$ or from eq. (B1) for $\rho_{0j} \propto \delta_{0j}$ and is given by

$$W(\Theta, E_2) \propto \frac{1}{\sin \Theta}.$$

Limitations of the validity of the expansion (13) arise when a correct description of the angular distribution is required for the entire range of angles and not only for

$$\sin \Theta \gtrsim \frac{1}{\min (I_{eff}, l_{eff})}.$$

The angular distribution of cascades of evaporated neutrons from very highly excited nuclei of very large angular momentum is obtained in the same way as that for a single emission process. Of the ν evaporated neutrons the first $(\nu-1)$ have an average kinetic energy of $2T$ at the nuclear surface, and consequently their angular momentum is small. Each of these neutrons has an angular distribution roughly given by eq. (16) with a usually rather small deviation from isotropy. For the last neutron, however, the situation is different. The available states of the residual nucleus have on the average

significantly lower spin values than before. The direction of the neutron angular momentum will therefore be more strongly correlated with the original spin, and the angular distribution of this neutron will approach the limit $1/\sin\Theta$. The decoupling of the total spin of the nucleus from its original direction and the change of its value due to the angular momenta of the emitted particles are negligible. It should be noted, however, that other modes of de-excitation, such as fission, which involve high angular momenta, will be greatly enhanced owing to the increased centrifugal barrier for the neutron and to the increasing probability of fission due to the rotation [3]). Therefore, the limiting distribution of the last neutron is important only when such modes can be neglected.

The following errors should be corrected:

Page 285, line 10 from bottom: $\hbar\Gamma_{\mathbf{I}}(\mathbf{1}, \mathbf{n}, E_2)$ should read: $\hbar^{-1}\Gamma_{\mathbf{I}}(\mathbf{1}, \mathbf{n}, E_2)$
Page 285, line 7 from bottom: $\hbar\Gamma_I$ should read: $\hbar^{-1}\Gamma_I$
Page 285, footnote : initial should read: compound nucleus
Page 285 eq. (3) : multiply the left-hand side by $\hbar^{-1}$ and the two right-hand sides by $1/4\pi$.
Page 286, line 7 : initial should read: compound nucleus
Page 287, eq. (10) : $Y_{nm}(\Theta, 0)$ should read: $Y_{nm}(\Theta, \varphi_0)$ (φ_0 is the angle between the x-axis and the projection of $\mathbf{n}$ on the (xy)-plane.)
Page 289, line 16 : l should read: min (I, l)
Page 289, eq. (21) : min $(I_{\text{eff}}, l_{\text{eff}})$ should read: max $(I_{\text{eff}}, l_{\text{eff}})$
Page 289, eq. (22) : $J_0(2i\alpha_t I(\sin\vartheta)$ should read: $J_0(2i\alpha_t Il\sin\vartheta)$
Page 292, eq. (B5) : $(-)^\lambda P_\lambda(\frac{1}{2}\pi)$ should read: $P_\lambda(0)$

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NEWS ITEMS

SYMPOSIUM ON CURRENT DEVELOPMENTS IN THE PRODUCTION OF HIGH VACUA

The Institute of Physics announces that it is arranging a one-day symposium entitled "Current developments in the production of high vacua" to take place in London on 17 April 1959.

There will be three sessions:

(a) Chemical and ionic pumping in kinetic vacuum systems.

(b) Problems in the production of high vacua in large equipment.

(c) Analysis of residual gases in kinetic vacuum systems.

Each of the first two sessions will be opened by invited speakers, and offers of short contributions, especially for the third session, will be considered by the Organizing Committee if outlines are submitted by 15 January 1959.

The symposium is intended mainly for persons working with high vacua. Abstracts (but not preprints) will be circulated before the symposium, the proceedings of which will not be published in full.

All communications regarding the symposium should be sent to the Secretary, The Institute of Physics, 47 Belgrave Square, London, S.W.1.

1959 CERN BUDGET APPROVED

At its Twelfth Session, held in Geneva on December 3rd, 1958, the Council of CERN approved contributions from the twelve Member States totalling 55 000 000 Swiss francs for 1959. This compares with a budget of 56 000 000 Swiss francs for 1958.

Following a discussion of the reports on 1958 activities submitted by Professor C. J. Bakker, Director-General of CERN, and by the Divisional Directors, the Council unanimously expressed its approval of the remarkable progress made during 1958 in the construction of CERN's two big particle accelerators and of CERN's other buildings and ancillary facilities.

The first machine, a 600 MeV synchro-cyclotron, came into operation on August 1st, 1957. It was recently used to perform a highly crucial experiment demonstrating, for the first time, the $e+\nu$ mode of decay of the π-meson — an experiment of critical importance to fundamental nuclear theory, which was announced last September at the "Atoms for Peace" Conference. The Council congratulated the Directors and members of the Synchro-cyclotron Division on this achievement, the first major research project completed in CERN.

In view of the rapid progress made at the CERN building site during 1958, the Council now feels confident that CERN's second accelerator, the 25 GeV proton synchrotron, will be finished in 1960, the target date originally set.

Finally, the election of new Officers of the Council was announced:

M. F. de Rose (France) was re-elected President of the Council. Likewise re-elected as Vice-Presidents were Professor W. Heisenberg (German Federal Republic) and M. J. Willems (Belgium).

Elected as new members to the Committee of Council were Professor P. Scherrer (Switzerland) and Dr. G. Funke (Sweden). The term of Sir Harry Melville, Secretary of the Department of Scientific and Industrial Research of the United Kingdom, as a member of the Committee, was also renewed.

M. J. H. Bannier (Netherlands) was re-elected Chairman of the Council's Finance Committee.

Professor E. Amaldi (Italy) remains Chairman of the Council's Scientific Policy Committee.

Vice-Chairman of the latter continues to be Professor L. Leprince-Ringuet (France). Re-elected as members of the Scientific Policy Committee were Professor W. Heisenberg (German Federal Republic) and Professor H. Alfvén (Sweden).

ELECTRON MICROSCOPE SOCIETY OF AMERICA

The next annual meeting will be held September 9—12, 1959 in Columbus, Ohio at the Ohio State University. Correspondence to D. Maxwell Teague, Secretary, Electron Microscope Society of America, P.O. Box 1118, Detroit 31, Michigan.

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BOOK REVIEWS

D. J. HUGHES, J. E. SANDERS and J. HOROWITZ (Editors), *Progress in Nuclear Energy*, Series I: Physics and Mathematics, Vol. 2 (Pergamon Press, London-New York-Paris-Los Angeles, 1958. viii-375 pages. 90 s.)

This is the second volume in the Physics & Mathematics series of the well known international review series, Progress in Nuclear Energy. Like the preceding volume, it contains contributions by a number of highly qualified specialists, who have reviewed their fields of interest in reactor physics. It consists of eight chapters covering as widely separated subjects as cross section data for the heavy elements and the application of Monte Carlo methods in transport problems. While the first volume in the series essentially contained a review and evaluation of data presented at the 1955 Geneva Conference this volume could serve as an introduction to some of the results presented at the Second Geneva Conference 1958. The two first chapters deal with neutron cross sections of heavy nuclei. The first, written by W. D. Allen and R. L. Henkel, covers the fast neutron data, while the second, by J. A. Harvey and R. B. Schwartz, deals with the thermal and resonance region. Together with the articles in the first volume on the same subject they probably constitute the best existing review of results in this special field.

The third and largest chapter deals with the fast developing subject of measurement technique and interpretation of reactor spectra; it is, I think, a general feeling that this subject hitherto has not received the attention that would justify the big emphasis which has continuously been put on the fundamental cross section measurements from the point of view of reactor technique. The chapter is competently written by M. J. Poole, M. S. Nelkin and R. S. Stone. At this point, the observation should be made that the authors are very fairly distributed on both sides of the Atlantic, and in some cases a British-American co-authorship warrants the presentation of all relevant viewpoints, as should be required of a review article in an international series.

Chapter four deals with the techniques of neutron source standardization, so important for the comparison and use of work made at different laboratories. The author is R. Richmond of Harwell. The more recently developed technique of making diffusion measurements with pulsed neutron sources, and especially the advantages of this method over the conventional one, are described with warmth by G. von Dardel and N. G. Sjöstrand in chapter five. The last three chapters deal with different aspects of the problem of reactor calculation. The controversial and complicated subject of the resonance escape probability in thermal reactors is dealt with in chapter six where both the Western and the Russian approach to the calculation of the resonance escape probability for heterogeneous reactors are expounded by J. B. Sampson and J. Cherwick. Since the advent of large and fast computing machines, new methods in the calculation of the critical size of reactors are possible; this is particularly true with respect to the multigroup calculations. A. Massit gives an account of the present methods used for calculation of heterogeneous reactors. In the last chapter, a general exposition of the use of Monte Carlo methods in the field of nuclear energy is given by G. Goertzel and M. H. Kalos, including both the more fundamental theory and some specific applications to reactor problems. In conclusion, it must be said that for those actively engaged in the physical problems of nuclear reactors this book will be very valuable as an up-to-date account of opinions of specialists in the different fields of nuclear reactor physics.

Carl F. Wande

W. MIALKI, *Kernverfahrenstechnik, eine Einführung für Ingenieure.* (Springer-Verlag, Berlin-Göttingen-Heildelberg 1958. x-472 pages. DM 25.50).

The book is written for students and engineers who want an introduction to the field of nuclear engineering. It covers all the main subjects related to nuclear reactors.

Nearly 30 per cent of the contents is reserved for a good account of basic reactor theory. As the book is intended for people who may not be familiar with physics it also contains a rather large section on basic physics. Like other books in the field, it does not contain any section on mathematics, especially on differential equations; such a section might be very helpful for many engineers. Sections on radiation protection and radiation detection are also provided, as well as sections on materials for reactors, manufacture and reprocessing of reactor fuel and heat transmission. It may be said that too little importance is attached to the technological problems related to reactors; e.g. the section on materials for reactors only occupies four per cent of the book.

The book is provided with a number of good figures and tables; in particular, a collection of tables on the last pages of the book will be a great help for the reader.

Torkil Hesselberg Jensen

M. HAÏSSINSKY (Editor), *Actions chimiques et biologiques des radiations* (*Chemical and biological actions of radiations*) Vol. 3 (Masson et Cie, Paris 1958. 222 pages. 4800 fr. fr.)

This volume of a well-known series contains three review articles concerned with the making and breaking of organic covalent bonds by radiation. The articles are (1) Radiolysis of organic liquids, by Milton Burton (transl. into French by Mme A. Bernas); (2) Polymerisations initiated by ionising radiations, by A. Chapiro and M. Magat; and (3) Effects of rays of large energy on polymers, by A. Charlesby (Transl. into French by Mme. C. Vermeil).

The first article starts with a theoretical consideration of bond energies in relation to the large range of chemical radiosensitivities observed. It then tabulates concisely many observations on degradation and polymerisation of low-molecular hydrocarbons, alcohols etc. and discusses them in relation to the theory. The second article describes how free radicals produced by radiation can initiate chain reactions, mostly vinyl polymerisations. There is also a little about the much less explored subject of polymerisations induced by ions instead of by free radicals, and on the fascinating subject of grafted polymers. The third article though concerned with polymers like the second, is more closely related to the first, being concerned with the breaking of bonds and subsequent cross-linkage or degradation in non-chain reactions. There is a satisfying blend of fundamental theory and measurements of the interesting and technically useful changes in physical properties of the polymers concerned. The book as a whole is a useful first-hand account of three interesting and closely related fields. English readers may regret having to retranslate two of the articles, but the translations are clear and easy to read.

J. St. L. Philpot

N. R. HANSON, *Patterns of discovery* (Cambridge University Press 1958. ix—241 p. 30s.)

The author writes entertainingly of gestalt-psychology and the behaviour of great scientists of the past; but when he comes to our own times, his ingenious patterns somehow forsake him, and he is unable to sort the wheat from the tares.

L.R.

L. D. LANDAU and E. M. LIFSCHITZ, *Quantum mechanics*, non-relativistic theory (Pergamon Press, London etc. 1958. xii—515 p. 80s.)

The Pergamon Press is doing a great service to students and teachers of physics by bringing out an English translation of Landau and Lifschitz' nine-volume course of theoretical physics, of which the seven volumes hitherto written have been instrumental in creating and upholding in the Soviet Union one of the most brilliant of the schools of theoretical physics flourishing at present at so many places in the world.

The authors do not belong to the generation of the founders of quantum mechanics, and their treatise has accordingly a different character from the great classics like Dirac's or Kramers' book. These are works of art, in which the author's personal outlook is strongly emphasized; the present treatise is also a work of art, but (if the simile may be pursued) of a more functionalistic type: it strives, with outstanding success, to give a well-balanced, systematic exposition of the subject in all its aspects. One cannot but admire almost at every page the thoughtful and skilful arrangement of the material as well as the wise selection of the most important features and the emphasis unfailingly put upon all essential points. At the same time, the scope of the book (though limited to non-relativistic phenomena) is very wide, including a full account of atoms, molecules and crystal symmetries, as well as the main features of collision theory. Everywhere the argument is developed with painstaking precision, yet concisely, in a clear, straightforward way, reminding one of Lorentz' best style. This style no longer conveys the thrill of the creative period, but in compensation it displays the harmony of the finished construction to the fullest. It is in perusing such a book as this that one vividly realizes how perfect in all its parts the conceptual structure of quantum mechanics is, and how firm its foundation.

L.R.

J. D. JACKSON, *The physics of elementary particles* (Princeton University Press 1958. x—135 p. $ 4.50)

This book contains the text of a course of lectures given at a Canadian summer school in 1957 (additions of recent results have been made until May 1958). It has the usual advantages and drawbacks of this kind of performance: the terseness and vivacity of the lecturing style on the one hand, the impossibility of achieving a completely systematic treatment on the other. Thus, we are here only presented with a selection from the many topics of interest concerning the fundamental constituents of matter (a phrase which would seem preferable to "elementary particles" to denote things which are neither particles nor elementary). The first 50 pages are devoted to an outline of pion physics, the next 40 pages to a survey of facts and speculations about K-mesons and hyperons, and the rest of the book to the consideration of decay processes from the point of view of symmetries and conservation laws. In the preface, the author expresses the modest hope that "this volume will serve for a short time as a useful introduction to a fascinating subject": we can certainly grant him that.

L.R.

A. M. WEINBERG and E. P. WIGNER, *The physical theory of neutron chain reactors* (University of Chicago Press 1958. xii—801 p. $ 15)

Here is a treatise on reactors of a quite different character from the scores of recent books dealing with this practical subject. While most other books are written at a technological level, or at least with the emphasis on the practical application of the procedures discussed, the work under review aims at a deep-going analysis, by rigorous scientific methods, of the various types of reactors. It is not intended for the training of technicians, but for the guidance of those whose task it is to solve the more difficult technological problems, requiring a thorough knowledge of the basic principles.

As one would expect, the distinguished authors have produced a masterpiece of lucid exposition of the intricate subject, deriving added authority from the fact that the methods developed in it have stood the test of long experience. The first quarter of the book is a brilliant compendium of the theory of nuclear reactions and of the fission process. The following broad subdivisions are successively devoted to the transport theory of neutrons, general reactor theory and the theory of heterogeneous reactors. In a vigorous preface, the authors express strong deprecation of excessive recourse to machine-computation and stress the need for a real understanding of the problems: a plea highly relevant also to other parts of nuclear physics than the study of reactors!

L.R.

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ERRATA

TORLEIF ERICSON, Level Density of Degenerate Fermi Systems,
(Nuclear Physics 8 (1958) 265).

p. 272, eq. (1.32): $g(\mu-\varepsilon_0)$ should read: $g_0(\mu-\varepsilon_0)$

p. 275, curves of figure 1: a should read: b
b should read: a

p. 278, eq. (2.4): the integrand should be multiplied by $e^{\beta\varepsilon}$

p. 279, line 9: $\rho_e(U+2\delta)$ should read: $\dfrac{2}{1+e^{\beta\delta}}\,\rho_e(U+2\delta)$

p. 280, eq. (A3): the extreme right-hand side should be multiplied by $\dfrac{1}{\sinh \pi\alpha}$

p. 281, eq. (A12), last line: $G(akt)$ should read: $F(akt)$

p. 281: eq. (A14) should read: $\det A \approx 4t^4 g_0^2 \Psi(0)$

p. 282, eq. (A16): the right-hand side should be multiplied by β^{-4}

p. 283, eq. (A17): the last two diagonal terms: $g_1(\mu_1+\varepsilon)+g_1(\mu_1-\varepsilon)$
and $g_2(\mu_2+\varepsilon)+ g_2(\mu_2-\varepsilon)$, should be in brackets.

S. M. Bilenky, L. I. Lapidus, L. D. Puzikov and R. M. Ryndin, Phenomenological Analysis of Reactions of the a+a′ → b+b′ Type, Nuclear Physics 7 (1958) 646.

On p. 648, eq. (9) should read

$$N_0(i_1, i_2, I_1, I_2) = \tfrac{1}{2}[(2i_1+1)(2i_2+1)(2I_1+1)(2I_2+1)\pm(-1)^{i_1+i_2+I_1+I_2}].$$

AUTHOR INDEX

ALMÉN, O., *see* NIJGH

ANDERSEN, S. L., H. BÖ, T. HOLTEBEKK, O. LÖNSJÖ and R. TANGEN, Resonance energies and half-widths for proton capture in aluminium 509

ARAKI, GENTARO, Quantal derivation of mass defect 650

ASCOLI, R. and E. MINARDI, On quantum theories with indefinite metric 242

BADRINATHAN, C., *see* KONDAIAH

BAKKE, F. H., Density of low-lying particle levels in odd-mass nuclei in the regions of large deformation 670

BALDIN, A. M., Electric polarizability of atomic nuclei 237

BARASHENKOV, V. S., V. A. BELYAKOV, E. G. BUBELEV, WANG SHOU FENG, V. M. MALTSEV, TEN GYN and K. D. TOLSTOV, Multiple production of particles in collisions between 9 GeV protons and nucleons 74

BARASHENKOV, V. S. and B. M. BARBASHOV, The electrical polarizability of the meson cloud in the nucleon 426

BARBASHOV, B. M., *see* BARASHENKOV

BAYMAN, B. F. and A. BOHR, On the connection between the cluster model and the SU_3 coupling scheme for particles in a harmonic oscillator potential 596

BECK, F., Einteilchen-Strahlungsübergänge im optischen Modell des Atomkerns 140

BELAYAKOV, V. A., *see* BARASHENKOV

BHATIA, A. B., *see* TRAINOR

BISWAS, S. N., *see* KUMAR

BÖ, H., *see* ANDERSEN

BODMER, A. R., Isotope shift and changes of nuclear radius 371

BOHR, AAGE and BEN R. MOTTELSON, Electric dipole moment associated with octupole vibrations of a spheroidal nucleus (Additional remarks) 687

BOHR, A., *see* BAYMAN

BOZÓKI, G. and EVA GOMBOSI, On the energy dependence of the anisotropy of high-energy nuclear interactions 400

BUBELEV, E. G., *see* BARASHENKOV

CHOU KUANG-CHAO, A note on the decay of the Σ-hyperon and its antiparticle 652

COESTER, F. and H. KÜMMEL, Time dependent theory of scattering of

nucleons by nuclei 225
DELVES, L. M., Tertiary and general-order collisions 391
D'ESPAGNAT, B. and J. PRENTKI, Global and restricted symmetry tests in antihyperon production 326
DŽELEPOW, B. S., YU. V. KHOLNOV and V. P. PRIKHODTSEVA, A $0^+ \rightarrow 0^+$ transition in Ce^{140} 665
EDEN, R. J., Series expansions and variational methods for the energy of a system of fermions 65
EL-NADI, M. and M. WAFIK, The characteristic level and the nuclear level density 22
ERICSON, T. and V. STRUTINSKY, On angular distributions in compound nucleus processes (Addendum) 689
EUBANK, H. P., R. A. PECK, Jr. and F. L. HASSLER, Proton distributions from $S^{32}(n, p)P^{32}$ 273
EVANS, J. E., B. B. KINSEY, J. R. WATERS and G. H. WILLIAMS, Radiation widths of neutron resonances in Ta 205
FERRERO, F., R. MALVANO, S. MENARDI and O. TERRACINI, The nuclear photoeffect in Al^{27} 32
FINK, R. W., *see* SCALAN
FIRK, F. W. K., Neutron resonance parameters of tantalum from 20 to 100 eV 198
FORD, KENNETH W. and E. J. KONOPINSKI, Evaluation of Slater integrals with harmonic oscillator wave functions 218
FREEMAN, JOAN M. and J. H. MONTAGUE, Gamma-rays from inelastic neutron scattering in Na^{23} 181
GATLAND, I. R., The effect of a scalar meson interaction between muons and baryons 267
GIRGIS, R. K., *see* YTHIER
GOLDANSKY, V. I., On the direct neutron exchange interaction of complex nuclei involving a possible large change of the nuclear spins 551
GOMBOSI, EVA, *see* BOZÓKI
GÜTTINGER, W., Non-local structure of quantized field theories of the second kind 429
HAGEDOORN, H. L., *see* KONIJN
HARRIS, I., *see* JASTROW
HASSLER, F. L., *see* EUBANK
HOLTEBEKK, T., *see* ANDERSEN
HUBY, R., M. Y. REFAI and G. R. SATCHLER, Predictions of the distorted wave theory of deuteron stripping reactions 94
HUIZENGA, J. R., *see* NIJGH
HUTCHINSON, G. W., *see* LAW

IYENGAR, K. V. K., *see* KONDAIAH
JASTROW, R. and I. HARRIS, Nuclear cross sections for the scattering of neutrons and protons 437
JOHANSSON, C. M. and I. P. K. LINDGREN, Spins of bismuth isotopes of masses 203, 204, 205 and 206 44
KANELLOPOULOS, TH., *see* WILDERMUTH
KASCHLUHN, FRANK, Dispersion relationl for pion-deuteron scattering (II) 347
KHOLNOV, YU. V., *see* DŽELEPOW
KINSEY, B. B., *see* EVANS
KJÄLLQUIST, KERSTIN, On the calculation of the Y_{40} deformation in strongly deformed heavy nuclei 163
KÖHLER, H. S., On the inelastic scattering of nucleons by nuclei at high energies 49
KOLCHINSKAYA, T. I. *see* SERGEYEV
KOMAR, A. A., Bremsstrahlung and pair production of particles of spin $\frac{3}{2}$ 6
KONDAIAH, E., C. BADRINATHAN and K. V. K. IYENGAR, Interaction of fast neutrons with fluorine 561
KONOPINSKI, E. J., *see* FORD
KONIJN, J., B. VAN NOOIJEN, H. L. HAGEDOORN and A. H. WAPSTRA, Branching ratios of electron capture to positon emission 296
KUMAR, KAILASH and S. N. BISWAS, Strange particle contribution to the magnetic moment of the nucleon 446
KÜMMEL, H., *see* COESTER
LATYSHEV, G. D., *see* SERGEYEV
LAW, MARGARET E., G. W. HUTCHINSON and D. H. WHITE, Total cross sections for 910 MeV protons 600
LINDGREN, I. P. K., *see* JOHANSSON
LÖNSJÖ, O., *see* ANDERSEN
MALTSEV, V. M., *see* BARASHENKOV
MALVANO, R., *see* FERRERO
MARIS, TH. A. J., Quasi-free nucleon-nucleon scattering 577
MARKLUND, INGVAR, Decay of Eu^{152m} 83
MARKLUND, INGVAR and LORNE A. PAGE, On the helicity of neutrinos 88
MARTY, N. et C. MARTY, Frédéric Joliot (1900—1958) 1
MARTY, C., *see* MARTY
MARX, G., On the second order wave equation of the fermions . . . 337
MENARDI, S., *see* FERRERO
MINARDI, E., *see* ASCOLI
MITTELSTAEDT, P., On the rearrangement energy of nuclear matter . 116
MONTAGUE, J. H., *see* FREEMAN

MOTTELSON, BEN R., *see* BOHR
NAGY, K. and J. ZIMANYI, Angular correlation between neutrino and gamma quantum released by electron capture 329
NIJGH, G. J., A. H. WAPSTRA, L. TH. M. ORNSTEIN, N. SALOMONS-GROBBEN, J. R. HUIZENGA and O. ALMÉN, Conversion coefficients of gamma transitions in ^{203}Tl 528
NIJGH, G. J. and A. H. WAPSTRA, Conversion coefficients for electric quadrupole transitions. 545
NIJGH, G. J., *see* WAPSTRA
ONEDA, S., Strange particle decays and the universal V—A four-fermion interaction 476
ORNSTEIN, L. TH. M., *see* NIJGH
ORNSTEIN, L. TH. M., *see* WAPSTRA
PAGE, LORNE A., *see* MARKLUND
PECK Jr., R. A., *see* EUBANK
PRADHAN, T., On the one-dimensional spinor model of Thirring . . 124
PRENTKI, J., *see* D'ESPAGNAT
PRIKHODTSEVA, V. P., *see* DŽELEPOW
REFAI, M. Y., *see* HUBY
REMENNY, A. S., *see* SERGEYEV
RICCI, R. A., *see* YTHIER
ROY, P. K., K^--absorption in nuclei, an isobaric spin and angular momentum analysis. 314
SALAM, ABDUS and J. TIOMNO, On the masses of elementary particles 585
SALOMONS-GROBBEN, N., *see* NIJGH
SALOMONS-GROBBEN, N., *see* WAPSTRA
SATCHLER, G. R., *see* HUBY
SCALAN, R. S. and R. W. FINK, Activation cross-sections for reactions of chlorine and copper with 14.8 MeV neutrons. 334
SERGEYEV, A. G., V. D. VOROBYEV, A. S. REMENNY, T. I. KOLCHINSKAYA, G. D. LATYSHEV and YU. S. YEGOROV, Influence of the finite dimensions of the nucleus on the relative conversion coefficients in the L-subshells 498
SITENKO, A. G., On the polarization of nucleons in high energy stripping reactions 412
SKYRME, T. H. R., The effective nuclear potential 615
SKYRME, T. H. R., The spin-orbit interaction in nuclei 635
SKYRME, T. H. R., Some distortion effects in the nuclear p-shell . . 641
SOKOLOV, A. A., Theory of Dirac particles with oriented spins and parity non-conservation 420
SOLOVIEV, V. G., On the superfluid state of the atomic nucleus . . . 655
SOOD, P. C., *see* TRAINOR

STRATHDEE, J. and Y. TAKAHASHI, Unrenormalizable interactions and the structure of elementary particles 558
STRUTINSKY, V., *see* ERICSON
TAKAHASHI, Y., *see* STRATHDEE
TANGEN, R., *see* ANDERSEN
TAYLOR, JOHN G., Possible parity assignments for the strange particles 357
TEN GYN, *see* BARASHENKOV
TER-MIKAYELIAN, M. L., On the theory of multiple scattering . . 679
TERRACINI, O., *see* FERRERO
TIOMNO, J., *see* SALAM
TOLSTOV, K. D., *see* BARASHENKOV
TRAINOR, L. E. H., A. B. BHATIA and P. C. SOOD, Quasi-lattice model of nuclear matter. 287
UHLMANN, A., Schema einer Quantenmechanik mit indefiniter Metrik 588
VAN LIESHOUT, R., *see* YTHIER
VAN NOOIJEN, B., *see* KONIJN
VERDE, MARIO, The inversion problem in wave mechanics and dispersion relations 255
VERVIER, J. F., Section efficace de capture radiative pour des neutrons d'une source Sb-Be. 569
VILLI, C., On a theorem concerning the single particle energy of a Fermi gas with interactions 306
VOROBYEV, V. D., *see* SERGEYEV
WAFIK, M., *see* EL-NADI
WANG SHOU FENG, *see* BARASHENKOV
WAPSTRA, A. H., Shape factors for first forbidden non-unique beta transitions 519
WAPSTRA, A. H., G. J. NIJGH, N. SALOMONS-GROBBEN and and L. TH. M. ORNSTEIN, Conversion coefficients of gamma transitions in ^{198}Hg 538
WAPSTRA, A. H., *see* KONIJN
WAPSTRA, A. H., *see* NIJGH
WATERS, J. R., *see* EVANS
WHITE, D. H., *see* LAW
WILDERMUTH, K. and TH. KANELLOPOULOS, On the structure of rotational levels of atomic nuclei 449
WILLIAMS, G. H. *see* EVANS
YEGOROV, YU. S., *see* SERGEYEV
YTHIER, C., R. K. GIRGIS, R. A. RICCI and R. VAN LIESHOUT, The decay of ^{73}Ga 108
ZIMANYI, J., *see* NAGY

SUBJECT INDEX

1. Nuclear Structure

1.C *Many-particle systems*

Series expansions and variational methods for the energy of a system of fermions, R. J. Eden 65
On the rearrangement energy of nuclear matter, P. Mittelstaedt 116
On a theorem concerning the single particle energy of a Fermi gas with interactions, C. Villi 306
The effective nuclear potential, T. H. R. Skyrme 615
The spin-orbit interaction in nuclei, T. H. R. Skyrme 635
Some distortion effects in the nuclear p-shell, T. H. R. Skyrme 641
On the superfluid state of the atomic nucleus, V. G. Soloviev 655

1.D *Nuclear models*

1.D.1 *Shell model*

Evaluation of Slater integrals with harmonic oscillator wave functions, Kenneth W. Ford and E. J. Konopinski 218
The effective nuclear potential, T. H. R. Skyrme 615
The spin-orbit interaction in nuclei, T. H. R. Skyrme 635
Some distortion effects in the nuclear p-shell, T. H. R. Skyrme . . . 641

1.D.2 *Collective model*

On the calculation of the Y_{40} deformation in strongly deformed heavy nuclei, Kerstin Kjällquist 163
Electric dipole moment associated with octupole vibrations of a spheroidal nucleus (Additional remarks), Aage Bohr and Ben R. Mottelson 687

1.D.3 *Other models*

Quasi-lattice model of nuclear matter, L. E. H. Trainor, A. B. Bhatia and P. C. Sood 287
On the structure of rotational levels of atomic nuclei, K. Wildermuth and Th. Kanellopoulos 449

1.D.4 *Relationship between different models*

On the connection between the cluster model and the SU_3 coupling scheme for particles in a harmonic oscillator potential, B. F. Bayman and A. Bohr 596

1.E *Properties of nuclei*

1.E.1 *Energy levels*

The characteristic level and the nuclear level density, M. El-Nadi and M. Wafik 22
Spins of bismuth isotopes of masses 203, 204, 205 and 206, C. M. Johansson and I. P. K. Lindgren 44
Decay of Eu^{152m}, Ingvar Marklund 83

Gamma-rays from inelastic neutron scattering in Na^{23}, Joan M. Freeman and J. H. Montague 181
Density of low-lying particle levels in odd-mass nuclei in the regions of large deformation, F. H. Bakke 670

1.E.2 *Binding energies; Masses*

Quantal derivation of mass defect, Gentaro Araki 650

1.E.5 *Nuclear radii; Matter distributions*

Isotope shift and changes of nuclear radius, A. R. Bodmer 371

2. Nuclear Reactions

2.A *General properties*

2.A.1 *Cross-sections, energy relations*

Neutron resonance parameters of tantalum from 20 to 100 eV, F. W. K. Firk 198
Radiation widths of neutron resonances in Ta, J. E. Evans, B. B. Kinsey, J. R. Waters and G. H. Williams. 205
Activation cross-sections for reactions of chlorine and copper with 14.8 MeV neutrons, R. S. Scalan and R. W. Fink 334
Interaction of fast neutrons with fluorine, E. Kondaiah, C. Badrinathan and K. V. K. Iyengar 561
Section efficace de capture radiative pour des neutrons d'une source Sb-Be, J. F. Vervier 569
Total cross sections for 910 MeV protons, Margaret E. Law, G. W. Hutchinson and D. H. White 600

2.A.2 *General theory*

Tertiary and general-order collisions, L. M. Delves 391

2.B *Angular distributions; Correlations; Polarization*

Proton distributions from S^{32} (n, p)P^{32}, H. P. Eubank, R. A. Peck, Jr. and F. L. Hassler 273
On the polarization of nucleons in high energy stripping reactions, A. G. Sitenko. 412
Nuclear cross sections for the scattering of neutrons and protons, R. Jastrow and I. Harris 437
On angular distributions in compound nucleus processes (Addendum), T. Ericson and V. Strutinsky 689

2.C *Resonance reactions; R-matrix theory*

Resonance energies and half-widths for proton capture in aluminium, S. L. Andersen, H. Bö, T. Holtebekk, O. Lönsjö and R. Tangen 509

2.D *Statistical model*

The characteristic level and the nuclear level density, M. El-Nadi and M. Wafik. 22

2.E *Optical model*

Einteilchen-Strahlungsübergänge im optischen Modell des Atomkerns, F. Beck 140

Time dependent theory of scattering of nucleons by nuclei, F. Coester and H. Kümmel 225
Nuclear cross sections for the scattering of neutrons and protons, R. Jastrow and I. Harris 437
On angular distributions in compound nucleus processes (Addendum), T. Ericson an V. Strutinsky 689

2.F *Direct interaction (other than stripping)*

Proton distributions from $S^{32}(n, p)P^{32}$, H. P. Eubank, R. A. Peck, Jr. and F. L. Hassler 273
On the direct neutron exchange interaction of complex nuclei involving a possible large change of the nuclear spins, V. I. Goldansky 551

2.G *Stripping and pick-up*

Predictions of the distorted wave theory of deuteron stripping reactions, R. Huby, M. Y. Refai and G. R. Satchler 94
On the polarization of nucleons in high energy stripping reactions, A. G. Sitenko 412

2.I *Photonuclear processes*

The nuclear photoeffect in Al^{27}, F. Ferrero, R. Malvano, S. Menardi and O. Terracini 32
Electric polarizability of atomic nuclei, A. M. Baldin 237

2.J *Fission and spallation*

Multiple production of particles in collisions between 9 GeV protons and nucleons, V. S. Barashenkov, V. A. Belyakov, E. G. Bubelev, Wang Shou Feng, V. M. Maltsev, Ten Gyn and K. D. Tolstov 74

2.L *Scattering processes (except two-nucleon system and meson scattering)*

On the inelastic scattering of nucleons by nuclei at high energies, H. S. Köhler 49
Gamma-rays from inelastic neutron scattering in Na^{23}, Joan M. Freeman and J. H. Montague 181
Time dependent theory of scattering of nucleons by nuclei, F. Coester and H. Kümmel 225
Nuclear cross sections for the scattering of neutrons and protons, R. Jastrow and I. Harris 437
Quasi-free nucleon-nucleon scattering, Th. A. J. Maris 577

3. Electromagnetic Radiation

3.A *Gamma radiation*

Decay of Eu^{152m}, Ingvar Marklund 33
Interaction of fast neutrons with fluorine, E. Kondaiah, C. Badrinathan and K. V. K. Iyengar 561
A $0^+ \rightarrow 0^+$ transition in Ce^{140}, B. S. Dželepow, Yu. V. Kholnov and V. P. Prikhodtseva 665

3.B *Internal conversion*

Influence of the finite dimensions of the nucleus on the relative conversion coefficients in the L-subshells, A. G. Sergeyev, V. D. Vorobyev, A. S. Remenny, T. I. Kolchinskaya, G. D. Latyshev and Yu. S. Yegorov . . 498

Conversion coefficients of gamma transitions in ^{203}Tl, G. J. Nijgh, A. H. Wapstra, L. Th. M. Ornstein, N. Salomons-Grobben, J. R. Huizenga and O. Almén 528
Conversion coefficients of gamma transitions in ^{199}Hg, A. H. Wapstra, G. J. Nijgh, N. Salomons-Grobben and L. Th. M. Ornstein 538
Conversion coefficients for electric quadrupole transitions, G. J. Nijgh and A. H. Wapstra 545

3.C *Extranuclear effects*

Bremsstrahlung and pair production of particles of spin $\frac{1}{2}$, A. A. Komar . . 6
Isotope shift and changes of nuclear radius, A. R. Bodmer 371

4. Beta Decay; Parity

4.C *Evaluation of matrix elements*

Shape factors for first forbidden non-unique beta transitions, A. H. Wapstra 519

4.D K-*capture*

Branching ratios of electron capture to positon emission, J. Konijn, B. van Nooyen, H. L. Hagedoorn and A. H. Wapstra 296
Angular correlation between neutrino and gamma quantum released by electron capture, K. Nagy and J. Zimanyi 329

4.E *Analysis of data*

The decay of ^{73}Ga, C. Ythier, R. K. Girgis, R. A. Ricci and R. van Lieshout 108
Shape factors for first forbidden non-unique beta transitions, A. H. Wapstra 519

7. Quantum Field Theory

7.A *General*

On the one-dimensional spinor model of Thirring, T. Pradhan 124
The inversion problem in wave mechanics and dispersion relations, Mario Verde 255
Dispersion relations for pion-deuteron scattering (II), Frank Kaschluhn . 347
Non-local structure of quantized field theories of the second kind, W. Güttinger 429

7.C π-*meson field* (*scattering and nuclear forces*)

The electrical polarizability of the meson cloud in the nucleon, V. S. Barashenkov and B. M. Barbashov 426

8. Elementary Particles

8.A *Experimental*

On the helicity of neutrinos, Ingvar Marklund and Lorne A. Page 88
On the energy dependence of the anisotropy of high-energy nuclear interactions, G. Bozóki and Eva Gombosi 400

8.B *Theory: interactions, transformation properties*

Bremsstrahlung and pair production of particles of spin $\frac{1}{2}$, A. A. Komar . 6
The effect of a scalar meson interaction between muons and baryons, I. R. Gatland 267

K^--absorption in nuclei, an isobaric spin and angular momentum analysis, P. K. Roy 314
Global and restricted symmetry tests in antihyperon production, B. d'Espagnat and J. Prentki 326
Angular correlation between neutrino and gamma quantum released by electron capture, K. Nagy and J. Zimanyi 329
Possible parity assignments for the strange particles, John G. Taylor . . 357
Theory of Dirac particles with oriented spins and parity non-conservation, A. A. Sokolov 420
Strange particle contribution to the magnetic moment of the nucleon, Kailash Kumar and S. N. Biswas 446
Strange particle decays and the universal V—A four-fermion interaction, S. Oneda 476
A note on the decay of the Σ-hyperon and its antiparticle, Chou Kuang-Chao 652

8.C *Fundamental theory of matter*

On quantum theories with indefinite metric, R. Ascoli and E. Minardi . . 242
On the second order wave equation of the fermions, G. Marx 337
Unrenormalizable interactions and the structure of elementary particles, J. Strathdee and Y. Takahashi. 558
On the masses of elementary particles, Abdus Salam and J. Tiomno . . . 585
Schema einer Quantenmechanik mit indefiniter Metrik, A. Uhlmann . . 588

9. Cosmic Radiation

9.B *Passage of particles through matter*

On the energy dependence of the anisotropy of high-energy nuclear interactions, G. Bozóki and Eva Gombosi 400
On the theory of multiple scattering, M. L. ter-Mikayelian 679

10. News Items

10.A *News items*

European Nuclear Energy Agency 179
Symposium on current developments in the production of high vacua . 691
1959 CERN budget approved 691
Electron Microscope Society of America 692

10.C *Obituaries*

Frédéric Joliot (1900—1958), N. et C. Marty 1

10.D *Book reviews*

ALLEN, J. S., The neutrino (L.R.) 178
Bevolkingsgroei en energie-verbruik (L.R.). 355
EISENBUD, L. and E. P. WIGNER, Nuclear structure (L.R.) 178
Elementary particle accelerators (L.R.) 355
GIBBS, R. C. and K. WAY, A directory to nuclear data tabulations (L.R.) 356
HAÏSSINSKY, M., editor, Actions chimiques et biologiques des radiations (Chemical and biological actions of radiations) Vol. 3 (J. St. L. Philpot) 694
HANSON, N. R. Patterns of discovery (L.R.) 694
HUGHES, D. J. *et al.*, editors, Progress in nuclear energy Series I: Physics and Mathematics, Vol. 2 (Carl F. Wendel) 693
JACKSON, J. D., The physics of elementary particles (L.R.) 695

LANDAU, L. D. and E. M. LIFSCHITZ, Quantum mechanics, non-relativistic theory (L.R.) 695
MIALKI, W., Kernverfahrenstechnik, eine Einführung für Ingenieure (Torkil Hesselberg Jensen) 694
Nuclear reactions in light nuclei (L.R.) 355
O.E.E.C., Applications of atomic science in agriculture and food (L.R.) . 355
PRIMAKOFF H. *et al.*, editors, Proceedings of the Midwest Conference on Theoretical Physics (L.R.) 356
SCHWARZSCHILD, M., Structure and evolution of the stars (L.R.) 178
SULLIVAN, W. H., Trilinear chart of nuclides, First revision sheet (L.R.) . 355
WEINBERG, A. M. and E. P. WIGNER, The physical theory of neutron chain reactors (L.R.) 696
WILSON, J. G. and S. A. WOUTHUYSEN, editors, Progress in elementary particle and cosmic ray physics, volume IV (L.R.) 178

10.E *Errata*

T. ERICSON and V. STRUTINSKY, Nuclear Physics **8** (1958) 284 690
T. ERICSON, Nuclear Physics **8** (1958) 265 697
S. M. BILENKY *et al.*, Nuclear Physics **7** (1958) 646 697

F. H. BAKKE, Density of low-lying particle levels in odd-mass nuclei in the regions of large deformation 670

M. L. TER-MIKAYELIAN, On the theory of multiple scattering 679

AAGE BOHR and BEN R. MOTTELSON, Electric dipole moment associated with octupole vibrations of a spheroidal nucleus (Additional remarks) 687

T. ERICSON and V. STRUTINSKY, On angular distributions in compound nucleus processes (Addendum) 689

News Items 691

Book Reviews 693

Errata 697

Author Index 698

Subject Index 703

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